Probability Distribution of Military Expenditure

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Abstract. The contribution deals with modeling of probability distribution of military expenditure. Analysis of relationship between military expenditure and other macroeconomic variables is topical in the defense economic literature. For this purpose, various models such as vector autoregressive, error correction or panel data models are applied. The probability distribution of analyzed variables can significantly influence the performance of estimated models. The aim of this paper is to find an appropriate probability distribution for data describing military expenditure per capita in the years 1993–2016. The statistical analysis involves data of 115 countries. The authors focus mainly on a normal, lognormal, gamma and Weibull distribution. The quality of a distribution fit is based on comparison of an empirical and a theoretical cumulative distribution function and is tested by Anderson-Darling, Cramer-von Mises and Kolmogorov-Smirnov test. In addition to the type of distribution, the development (changes) of the parameters estimates is discussed and the most sensitive test statistics is suggested. The authors compared the quantiles of the estimated probability distributions as potential risk indicators.

Keywords: distribution fitting, military expenditure, risk

JEL classification: C12, E69
AMS classification: 62F99

1 Introduction

Statistical analysis of relationship between military expenditure and other macroeconomic variables is very topical in the defense economic literature. For this purpose, various models such as vector autoregressive, error correction or panel data models are applied, see for example[10] or [6]. The probability distribution of analyzed variables can significantly influence the performance of estimated models. Normality of variables is a common and frequent assumption of many widely used models and methods. There are many normality test available whose properties differ [11]. Violation of the normality assumption can appreciably affect parameter estimates, individual tests associated with the estimated model. The problem of a probability distribution is addressed by a number of authors, for example the probability distribution of income is discussed in [2], distribution fitting of a microbiological contamination data is solved in [3], or economic growth is modeled in [12].

In this contribution, we focus on distribution of military expenditure (per capita). We would like to show that the probability distribution of military spending is not normal and should be modeled by another probability distribution, for example by a log-normal distribution. From a security perspective, it is useful to focus on the extreme values. We will pay attention to the small and, above all, high values of military spending that can be a marker of potential security risk. For this reason, we calculate the necessary quantiles of probability distributions used to model the distribution of military expenditure.

We analyze data from 1993 to 2016. According to SIPRI (Stockholm International Peace Research Institute), military expenditure include all current and capital expenditure on the armed forces (including peace keeping forces), defense ministries and other government agencies engaged in defense projects, paramilitary forces when judged to be trained, equipped and available for military operations, and military space activities. The amount of military spending in different countries varies considerably. Israel, Saudi Arabia, Oman, USA, Singapore, Kuwait, Norway Bahrain and Australia belong to countries with the largest military expenditure per capita in 2016 (military spending exceeds \$ 1,000). On the other end of the scale are countries like Malawi, Madagascar, Liberia, Mozambique, Sierra Leone or Ethiopia with military expenditure less than \$ 5 per head. The empirical distribution of military spending in not symmetric, it is skewed. A normal distribution is not a good model for this variable, as will be stated later.

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2 Data Description

As mentioned in the introduction, the data represent military expenditure of selected countries per capita in years 1993–2016. The dataset used for the study carried out in this paper comes from the SIPRI database of military expenditure, that is available online at: `https://www.sipri.org/databases/milex`. Figure 1 shows military expenditure data of selected countries.

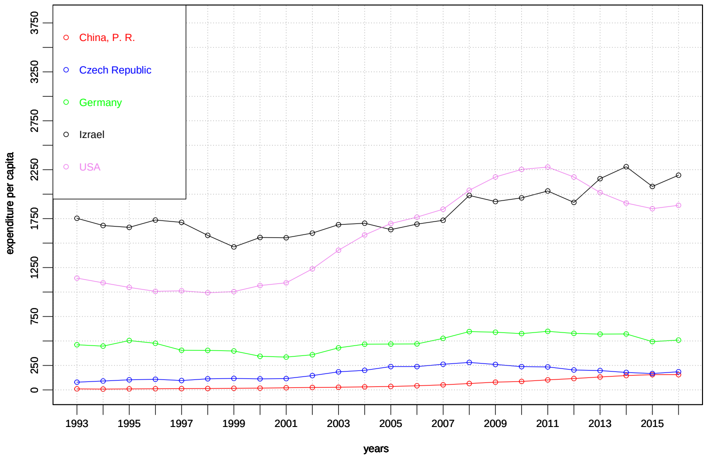


Figure 1 Military expenditure of China, the Czech Republic, Germany, Israel and USA from 1993 to 2016 (in dollars per capita)

For the purpose of the paper, from the original dataset obtained from SIPRI a selection was made. It is based on the relevance of countries included in the model and trustworthiness of the respective data. After selection the dataset contains information on 115 countries including the most prominent actors on the field of international politics like USA, China, Russian Federation, India, etc. as well as countries of former eastern block and countries of Middle East and Africa.

The time period of the data was set between the years 1993 and 2016, where the reason for the choice of the lower bound comes from the fact that data of many countries relevant for this study are not available prior to the year 1993. The upper bound corresponds to the most recent year for which the data is currently available in the SIPRI database. The goal of this paper is to study the probability distribution of military expenditure per capita in the above mentioned period by finding an appropriate distribution for military spending in each year. In case that a data entry was missing for a given country in a given year, that the country was not included into the data sample of military expenditure in that year.

Following notation will be used henceforth. Denote $n = 115$ the total number countries considered in this study and m the number of countries with missing entries in a given year j. Than $\boldsymbol{X}_j = (X_{1,j}, ..., X_{n-m,j})$ is a random sample of a size $n - m$ of military expenditure of countries in a year j.

3 Methods and Tests

It was assumed that the data sample $\boldsymbol{X}_j = (X_{1,j}, ..., X_{n-m,j})$ of military expenses of countries in a given year j had come from one of the following distributions: normal, lognormal, gamma and two-parameter Weibull distribution. For further information see [1]. The parameters μ_N, σ_N^2 of the normal distribution, μ_{LN}, σ_{LN}^2 of the lognormal distribution, the shape parameter α, the rate parameter β of the gamma distribution, the shape parameter k and the scale parameter λ of the Weibull distribution were estimated from the sample for each year j. Through the text the estimates of the parameters are denoted by hat symbol, e. g. $\hat{\mu}$ is the estimate of the parameter μ.

To obtain these estimates, maximum goodness of fit estimation as described in [8] was used, where the applied metric was Kolmogorov-Smirnov [4]. The choice of the maximum goodness of fit estimation using the Kolmogorov-Smirnov metric was done to avoid numerical computation problems that would occur while estimating parameters of the Weibull and gamma distribution when using Cramer-von Mises metric or Anderson-Darling metric in the maximum goodness of fit approach, or while using the maximum likelihood estimation approach.

Assume that the sample $\boldsymbol{X}_j = (X_{1,j}, ..., X_{n-m,j})$ comes from distribution with the distribution function $F_{\boldsymbol{X}_j}$

The following four hypotheses were tested for $j \in \{1993, 1994, \dots, 2016\}$ and the level of significance $\alpha = 0.05$:

$$H_{01}: \quad F_{\mathbf{X}_j} \text{ is equal to } F_{N(\hat{\mu}_{N,j}, \hat{\sigma}^2_{N,j})}, \tag{1}$$

$$H_{02}: \quad F_{\mathbf{X}_j} \text{ is equal to } F_{LN(\hat{\mu}_{LN,j}, \hat{\sigma}^2_{LN,j})}, \tag{2}$$

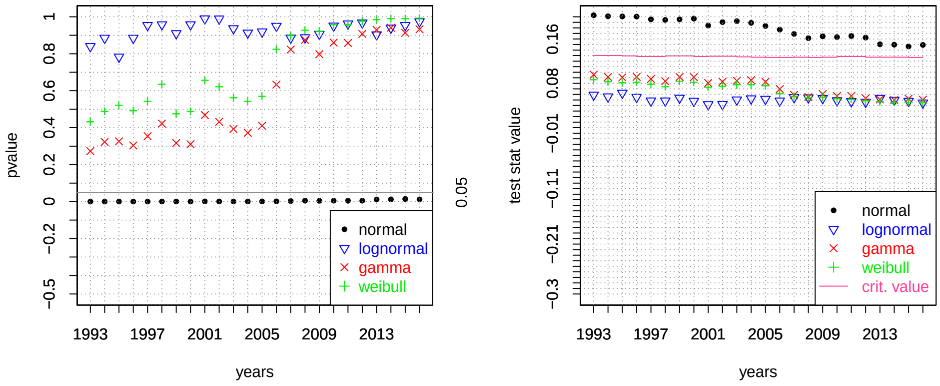
$$H_{03}: \quad F_{\mathbf{X}_j} \text{ is equal to } F_{G(\hat{\alpha}_j, \hat{\beta}_j)}, \tag{3}$$

$$H_{04}: \quad F_{\mathbf{X}_j} \text{ is equal to } F_{W(\hat{k}_j, \hat{\lambda}_j)}, \tag{4}$$

where $F_{N(\hat{\mu}_{N,j}, \hat{\sigma}^2_{N,j})}$, $F_{LN(\hat{\mu}_{LN,j}, \hat{\sigma}^2_{LN,j})}$, $F_{G(\hat{\alpha}_j, \hat{\beta}_j)}$ and $F_{W(\hat{k}_j, \hat{\lambda}_j)}$ denote in sequence the cumulative distribution functions of normal, lognormal, gamma and Weibull distributions, with the respective parameter estimates taken as the values of the actual parameters. The hypothesis testing was performed via tests based on comparing the empirical cumulative distribution function with the theoretical cumulative distribution function of the corresponding distributions using Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling metrics [4].

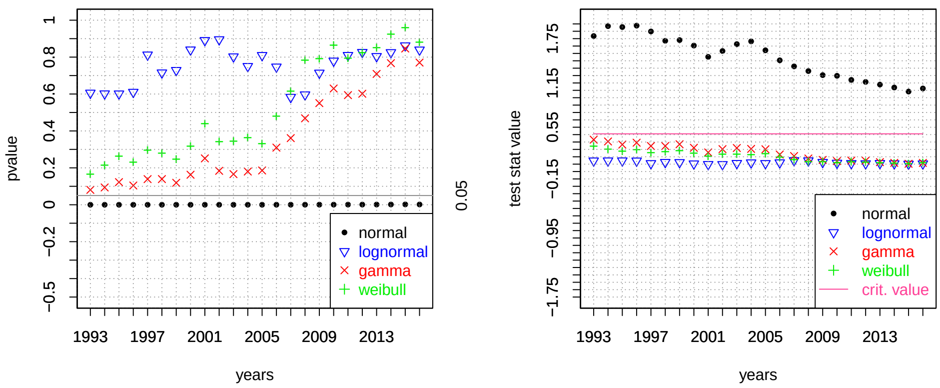
4 Results

In Figures 2a, 3a and 4a one can see how the p-values of the Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling test change for the years varying from 1993 to 2016. The gray horizontal line denotes the level of significance $\alpha = 0.05$. Figures 2b, 3b and 4b show how the test statistic values of the Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling test change in years 1993–2016. The violet-red step function displays the critical values of the test statistic for each year [5, 7, 9].



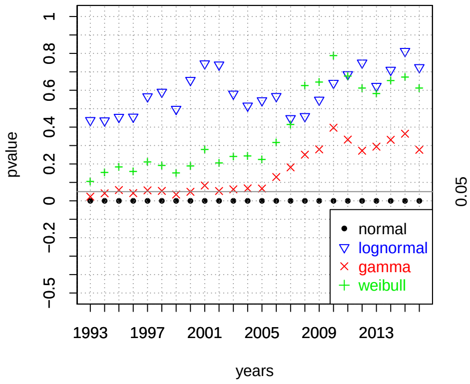
(a) P-values of Kolmogorov-Smirnov test **(b)** Values of the Kolmogorov-Smirnov test stat.

Figure 2 P-values and values of the Kolmogorov-Smirnov test statistic performed for years 1993–2016

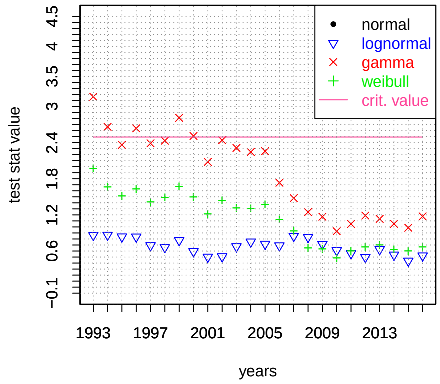


(a) P-values of Cramer-von Mises test **(b)** Values of the Cramer-von Mises test statistic

Figure 3 P-values and values of the Cramer-von Mises test statistic performed for years 1993–2016



(a) P-values of Anderson-Darling test



(b) Values of the Anderson-Darling test statistic

Figure 4 P-values and values of the Anderson-Darling test statistic performed for years 1993–2016

One can observe that the p-values of neither of the tests performed to test the hypothesis (1) attain values greater than the level of significance, hence in all considered years we reject the hypothesis (1). On the other hand all of the performed tests suggest, that the best fit for each year of the studied period provides the lognormal distribution.

We may notice in Figure 4a that the p-values of the Anderson-Darling test, when testing the hypothesis (3) for the years from 1993 to 2005, are close to the level of significance $\alpha = 0.05$. Furthermore, it can be observed in Figure 4b that some of the values of the Anderson-Darling test statistic in this time period lie above the values of the critical function in the given year. A closer look at the p-values of the Anderson-Darling test of the hypothesis (3) and the values of the respective test statistic and the critical values up to year 2000 are collected in Table 1. The year 2000 is the last one when the hypothesis (3) was rejected. It is clear from Table 1 that the hypothesis (3)

year	1993	1994	1995	1996	1997	1998	1999	2000
p-value	0.0227	0.0408	0.0586	0.0420	0.0568	0.0538	0.0342	0.0490
test static	3.1631	2.6627	2.3637	2.6384	2.3894	2.4335	2.8130	2.5113
critical value	2.4941	2.4941	2.4941	2.4940	2.4940	2.4941	2.4941	2.4940

Table 1 Anderson-Darling test results for years from 1993 to 2000, years when the hypothesis (3) is rejected are in red

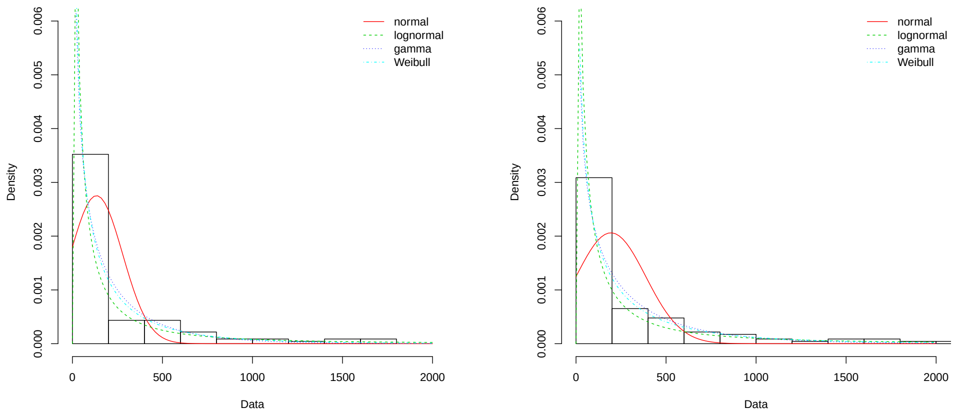
tested by the Anderson-Darling test is rejected for years 1993, 1994, 1996, 1999 and 2000 (denoted by red).

It should be noted that in general the p-value of the Kolmogorov-Smirnov test, when testing a given hypothesis for a given year, is greater than the p-value of the Cramer-von Mises test for the corresponding hypothesis and year, which is in turn greater than the p-value of the Anderson-Darling test (Figures 2a, 3a and 4a). We thus conclude that the Anderson-Darling test appears to be the most strict one for this dataset.

In all Figures 2, 3 and 4 one can notice that the fit of the gamma and Weibull distributions is initially poor for years from 1993 to 2005, but then significantly improves in years 2006–2008 to be almost on par with the fit of the lognormal distribution. This suggests that some change in the distribution of military expenditure has happened between years 2005 and 2008. When comparing the empirical densities in the respective years (see Figure 5) it

probabilities	0.01	0.05	0.10	0.90	0.95	0.99
quantiles – normal	−222.7324	−103.7562	−40.3305	407.1395	470.5653	589.5415
quantiles – lognormal	2.1000	6.7226	12.5004	994.1412	1848.5634	5917.7717
quantiles – gamma	0.1698	2.3652	7.4221	604.7310	834.7128	1390.1139
quantiles – Weibull	0.2975	2.9827	8.2556	647.5655	939.5561	1725.9847
empirical – quantiles	2.5909	5.5824	9.5225	736.3058	1287.5864	1974.7292

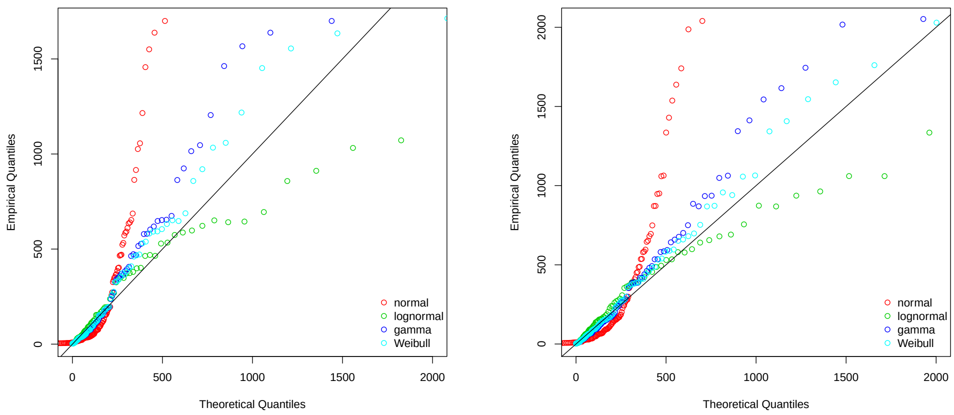
Table 2 Selected quantiles of the fitted distributions and the empirical quantiles in 2016



(a) Comparison of theoretical and empirical densities of distributions for year 2005

(b) Comparison of theoretical and empirical densities of distributions for year 2008

Figure 5 Comparison of theoretical and empirical densities of normal, lognormal, gamma and Weibull distributions



(a) Q-Q plot for year 2005

(b) Q-Q plot for year 2008

Figure 6 Q-Q plots of theoretical quantiles of normal, lognormal, gamma and Weibull distribution and empirical quantiles of the sample

may be noticed that military expenditure of countries experienced a slight increase in years between 2005 and 2008, most noticeably among countries, whose spendings were rather small prior to the year 2005. This is further supported by the Q-Q plots depicted in Figure 6.

Table 2 contains quantiles of estimated distributions and empirical quantiles of military expenditure in 2016. It was shown in the previous part of the contribution that the normal distribution is not an acceptable model for analyzed data. We come to the same conclusion when comparing the quantiles of the normal distribution with empirical quantiles or quantiles of other studied distributions. Hence, the quantiles of the normal distribution are also included in the table 2. One can observe from this comparison the mismatch between the empirical and the normal distribution at its tails. This further support the point that normal distribution is not suitable for modeling the data. If one analyzes quantiles of the gamma and Weibull distribution, they appear to be more or less comparable. Quantiles of the lognormal distribution are substantially different. They are close to the empirical quantiles for small probabilities, but their values for probabilities 0.95 and 0.99 are considerably greater. Estimates of quantiles for high probabilities can be taken as a marker of potential security risk. If military expenditure exceeds given value (say 95% quantile), one can deduce that military spendings in such a country are large (extreme) compared to the rest of the world. The choice of proper probability distribution strongly influences these estimates.

The ability to accurately describe the probability distribution of military expenditure, especially its high values, is useful from a security point of view. The empirical distribution suggests that military spending is expected to exceed \$1287.6 in 5% of countries. The corresponding quantiles of estimated distributions differ significantly from this value. For example, 95% quantile of the log-normal distribution has a value of \$1845.6, 95% quantile of Weibull distribution is \$939.6. The description of the probability distribution of military expenditure using selected (parametric) models is not accurate enough in the area of higher values. A possible solution to this problem can be the use of nonparametric estimation methods, a topic we would like to focus on in the future.

5 Conclusion

The topic of this contribution was to find a suitable probability distribution of military expenditure per capita from 1993 to 2016. The statistical analysis involves data of 115 countries. The authors focus on a normal, lognormal, gamma and Weibull distribution. The quality of a distribution fit is based on comparison of an empirical and a theoretical cumulative distribution function and was tested by Anderson-Darling, Cramer-von Mises and Kolmogorov-Smirnov test. The distribution of the analyzed variable is not symmetric, so the normal distribution was rejected by all tests. According to the test results, the best fit is obtained by using the lognormal model. A more detailed view of the quantiles showed that there are differences between the probability distributions. The models based on the gamma and Weibull distribution were similar in terms of estimated quantiles. Quantiles of the lognormal distribution model are substantially different. They are close to the empirical quantiles for small probabilities (1%, 5% quantiles), but their values for 0.95 and 0.99 are considerably greater. The lognormal distribution seems to be a reasonable model for military expenditure per head. However, it should be noted that the ability of this probability distribution model to describe extreme values (large values of military spending indicating a potential security risk) is not sufficient.

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