1 CONTINUOUS RANDOM VARIABLES

1.1 Uniform distribution

1.1.1 Exercises 1.1

1. Assume $X \sim R(\alpha, \beta)$. Prove
   a) $E(X) = \frac{\alpha+\beta}{2}$,
   b) $D(X) = \frac{1}{12}(\beta - \alpha)^2$,
   c) $x_P = \alpha + P(\beta - \alpha)$,
   d) $Me(X) = \frac{\alpha+\beta}{2}$.
2. The distribution function of a random variable X is:

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leq -4, \\ \dfrac{x+4}{6} & \quad -4 < x < 2, \\ 1 & \quad x \geq 2. \end{cases}$$

   a) Determine the probability density function, the mean, the variance, the median and the 30% quantile of the random variable X.
   b) Compute the probability that the random variable X assumes values greater than -3, smaller than 0, between -1 and 1.
3. Consider a bus line where busses come regularly every 15 minutes. A randomly chosen traveller comes to a bus stop. We observe a random variable X which represents the waiting time for the bus.
   a) Describe the random variable by means of the probability density function and the distribution function (including graphs).
   b) Compute the probability that randomly chosen traveller waits most highly 5 minutes, exactly 10 minutes, at least 3 minutes, between 3 and 10 minutes.
   c) Determine the mean, the median, the variance, the standard deviation and 90% quantile of the random variable X.
4. A delivery service is supplied randomly from 7 to 9 a.m.. We observe a random variable X which represents the waiting time for the delivery.
   a) Describe the random variable by means of the probability density function and the distribution function (including graphs).
   b) Compute the probability that we must wait for most highly 40 minutes, exactly 1 hour, at least half an hour, at least 20 minutes, most highly 80 minutes.
   c) Determine the mean, the median, the variance, the standard deviation and 20% quantile of the random variable X.
5. We do not have a weight unit smaller than 1 g (If the outcome of a weighing is somewhere between 15 and 16 g, estimate the weight as 15.5 g). We observe a random variable X (with a continuous probability density function in the interval -0.5 to 0.5 g) which represents the random error during the weighing.

a) Describe the random variable by means of the probability density function, the distribution function, the mean and the variance of the random variable X.
b) What is the probability that the absolute error is not greater than 0.3 g?

Solution.
2. a) $f(x)$: 1/6 for $-4 < x < 2$; 0 otherwise; -1; 3; -1; -2.2; b) 0.833; 0.667; 0.333;
3. a) $f(x)$: 1/15 for $0 < x < 15$; 0 otherwise; $F(x)$: 0 for $x \leq 0$; $x/15$ for $0 < x < 15$; 1 for $x \geq 15$; b) 0.333; 0; 0.8; 0.467; c) 7.5; 7.5; 18.75; 4.330; 13.5;
4. a) $f(x)$: 1/2 for $0 < x < 2$; 0 otherwise; $F(x)$: 0 for $x \leq 0$; $x/2$ for $0 < x < 2$; 1 for $x \geq 2$; b) 0.333; 0; 0.75; 0.5; c) 1; 1; 0.333; 0.577; 0.4;
5. a) $f(x)$: 1 for $-0.5 < x < 0.5$; 0 otherwise; $F(x)$: 0 for $x \leq -0.5$; $x + 0.5$ for $-0.5 < x < 0.5$; 1 for $x \geq 0.5$; 0; 0.083; b) 0.6.

1.2 Exponential distribution

1.2.1 Exercises 1.2

1. Assume $X \sim E(\alpha, \delta)$. Prove
   a) $F(x) = 1 - e^{-\frac{x-\alpha}{\delta}}$ for $x > \alpha$,
   b) $D(X) = \delta^2$,
   c) $x_P = \alpha - \delta \ln(1 - P)$.
2. The probability density function of a random variable X is:

$$f(x) = \begin{cases} \dfrac{1}{100} e^{-\frac{x}{100}} & \text{for} \quad x > 0, \\ 0 & \phantom{\text{for}} \quad x \leq 0. \end{cases}$$

   a) Determine the distribution function, the mean, the standard deviation, the median and the 95% quantile of the random variable X.
   b) Compute the probability that the random variable X assumes values greater than 120, smaller than 75, between 75 and 120.
3. The working life of given bulbs is, on an average, 2000 hours. Let X be a random variable that represents the the working life, consider X has exponential distribution.
   a) Describe the random variable by means of the probability density function and the distribution function.
   b) Compute the probability that the bulb has the working life most highly 1000 hours, at least 2500 hours and between 1000 and 2500 hours.
   c) Determine the mean, the standard deviation, the median and the 90% quantile of the random variable X.

4. Assume a random variable X which represents the time between arrivals of lorries to a construction site, consider X has exponential distribution. The minimum time is 5 minutes, the average time is 10 minutes.
   a) Describe the random variable by means of the probability density function and the distribution function.
   b) What is the probability that the time between arrivals is most highly 7 minutes, at least 11 minutes, between 7 and 11 minutes?
   c) Determine the mean, the standard deviation, the median and the 20% quantile of the random variable X.

5. Assume a random variable X which represents the time when a client is served in a bank, consider X has exponential distribution with parameter $\alpha = 1$ minute. The probability that X is smaller than 7 minutes is 0.4682. What is the mean of the random variable X?

Solution.
2. a) $F(x)$: $1-e^{-x/100}$ for $x > 0$; 0 for $x \leq 0$; 100; 100; 69.315; 299.573; b) 0.301; 0.528; 0.171;
3. a) $f(x)$: $\frac{1}{2}e^{-x/2}$ for $x > 0$; 0 for $x \leq 0$; $F(x)$: $1 - e^{-x/2}$ for $x > 0$; 0 for $x \leq 0$; b) 0.393; 0.287; 0.320; c) 2; 2; 1.386; 4.605;
4. a) $f(x)$: $\frac{1}{5}e^{-(x-5)/5}$ pro $x > 5$; 0 for $x \leq 5$; $F(x)$: $1-e^{-(x-5)/5}$ for $x > 5$; 0 for $x \leq 5$; b) 0.330; 0.301; 0.369; c) 10; 5; 8.466; 6.116;
5. 10.5 minutes.

2. a) $F(x)$: $1-e^{-x/100}$ for $x > 0$; 0 for $x \leq 0$; 100; 100; 69.315; 299.573; b) 0.301; 0.528; 0.171;
3. a) $f(x)$: $\frac{1}{2}e^{-x/2}$ for $x > 0$; 0 for $x \leq 0$; $F(x)$: $1 - e^{-x/2}$ for $x > 0$; 0 for $x \leq 0$; b) 0.393; 0.287; 0.320; c) 2; 2; 1.386; 4.605;
4. a) $f(x)$: $\frac{1}{5}e^{-(x-5)/5}$ pro $x > 5$; 0 for $x \leq 5$; $F(x)$: $1-e^{-(x-5)/5}$ for $x > 5$; 0 for $x \leq 5$; b) 0.330; 0.301; 0.369; c) 10; 5; 8.466; 6.116;
5. 10.5 minutes.

1.3 Normal distribution

1.4 Standard normal distribution

1.4.1 Exercises 1.3 a 1.4

1. Assume $X \sim N(\mu, \sigma^2)$. Derive the formula $x_P = \mu + \sigma \cdot u_P$.
2. $X \sim N(0, 1)$, derive from tables: $\Phi(1.57)$, $\Phi(-2.25)$, $u_{0.975}$, $u_{0.10}$.
3. Compute the probability that a random variable $X \sim N(\mu, \sigma^2)$ occurs in the interval a) $(\mu - \sigma, \mu + \sigma)$, b) $(\mu - 2\sigma, \mu + 2\sigma)$. Interpret results.

4. A measuring is attended by random errors. The meter is is under the influence of a constant error 0.5 m. Let X be a continuous random variable which represents random errors, consider X has normal distribution with the standard deviation 0.3 m.
   a) Find the probability that the absolute value of the random error is most highly 1 m.
   b) Find the value of the random variable which is not exceeded with the probability 0.90.
5. Assume a continuous random variable X which represents the time which is needed to complete a test, let X has a normal distribution with $\mu = 50$ minutes and $\sigma = 10$ minutes.
   a) How many students finish the test within an hour?
   b) Find the time which correspond to the fact that at lest 90 % of students finish the test.
6. A guarantee period is 2 years for all products of a company. The profit from one product is constantly 520 Eur, the loss from each item under claim is 1000 Eur. Assume a continuous random variable X which represents the lifetime of the product, let X has a normal distribution with $\mu = 5.6$ years and $\sigma = 1.7$ years (most highly one claim is possible).
   a) Find the probability that the product must be claimed and find the average profit from a sold product.
   b) Find the change of the guarantee period so that the profit is at least 508 Eur.

Solution.
2. 0.94179; 0.01222; 1.960; −1.282;
3. a) 0.683; b) 0.954;
4. a) 0.953; b) 0.884;
5. a) 84.1; b) 63;
6. a) 0.017; 503; b) 21 months.

1.5 Log-normal distribution

1.5.1 Exercises 1.5

1. Assume $X \sim LN(\mu, \sigma^2)$. Derive the formula $x_P = e^{\mu + \sigma u_P}$.
2. Let a random variable X has a log-normal distribution $LN(3.5; 0.36)$, denote $F(x)$ as a corresponding distribution function.
   a) Determine $F(16)$, the mean, the standard deviation, the mode, the 5% quantile and the skewness of the random variable X.
   b) Find the probability that the random variable X is smaller than 20, greater than 30, between 20 and 30? Comment the sum of these two probabilities.

3. Consider the size of elements (unit mm) of given sand is a random variable X. Let X has a log-normal distribution with parameters $\mu = 2.5$ and $\sigma^2 = 0.16$.
   a) Find the proportion of elements with the size 10–15 mm in given sand?
   b) Determine the mean, the standard deviation, the median, the skewness and the kurtosis of the random variable X.
4. It is known that the time to repair of given device (unit hour) has a log-normal distribution with parameters $\mu = 2.3$ and $\sigma^2 = 0.64$.
   a) Find the probability that the time to repair is at least 15 minutes.
   b) Determine the mean, the variance, the standard deviation, the median and the mode of the random variable X.
5. Assume a distance between vehicles on a highway (unit second) is a random variable X which has a log-normal distribution with parameters $\mu = 1.27$ and $\sigma^2 = 0.49$.
   a) Find the proportion of distances 4–5 seconds?
   b) Determine the mean, the variance, the standard deviation, the median, the mode and the skewness of the random variable X.

2. a) 0.113; 39.646; 26.,098; 23.104; 12.343; 2.260; b) 0.200; 0.564; 0.236;
3. a) 38.6 %; b) 13.197; 5.497; 12.182; 1.322; 3.260;
4. a) 0.305; b) 13.736; 169.139; 13.005; 9.974; 5.259;
5. a) 11.7 %; b) 4.549; 13.087; 3.618; 3.561; 2.181; 2.888.

1.6 Student's t-distribution, chi-square distribution, Fisher-Snedecor distribution

1.6.1 Exercises 1.6

1. Let random variable t has a Student's distribution $t(\nu)$.
   a) Find the 99% quantile for $\nu = 4$ and 23.
   b) Find the 2.5% quantile for $\nu = 5$ and 27.
   c) Find approximately values of quantiles $t_{0.10}(45)$ and $t_{0.95}(126)$.
2. Let random variable t has a Student's distribution $t(24)$.
   a) Find 2.5% and 97.5% quantiles of the random variable t.
   b) Compute $P(t > -2.064)$.
3. Let random variable χ^2 has a chi-square distribution $\chi^2(\nu)$.
   a) Find the 95% quantile for $\nu = 3$ a 26.
   b) Find the 10% quantile for $\nu = 6$ a 24.
   c) Find approximately values of quantiles $\chi^2_{0.975}(80)$ and $\chi^2_{0.05}(120)$.
4. Let random variable χ^2 has a chi-square distribution $\chi^2(15)$.
   a) Find 5% a 95% quantiles of the random variable χ^2.
   b) Compute $P(\chi^2 < 7.26)$.

5. Let random variable F has a Fisher-Snedecor distribution $F(\nu_1, \nu_2)$.
   a) Find $F_{0.95}(8, 15)$ and $F_{0.975}(9, 4)$.
   b) Find $F_{0.05}(22, 4)$ and $F_{0.025}(10, 20)$.
   c) Find $F_{0.95}(55, 40)$ and $F_{0.975}(30, 34)$.
6. Let random variable F has a Fisher-Snedecor distribution $F(12, 7)$.
   a) Find 5% a 95% quantiles of the random variable F.
   b) Compute $P(F < 4.666)$.

Solution.

1. a) 3.747; 2.500; b) −2.571; −2.052; c) −1.282; 1.645;
2. a) −2.064; 2.064; b) 0.975;
3. a) 7.81; 38.9; b) 2.20; 15.7; c) 106.14; 95.42;
4. a) 7.26; 25.0; b) 0.05;
5. a) 2.641; 8.905; b) 0.355; 0.292; c) (1.637; 1.693); (1.943; 2.074);
6. a) 0.343; 3.575; b) 0.975.