1 PROBABILITY

1.1 Basic Combinatorics

1.1.1 Exercises 1.1

1. Simplify
$$\binom{4}{4}+\binom{5}{4}+\binom{6}{4}+\binom{7}{4}+\binom{8}{4}.$$
2. Solve the equation:
$$\binom{x}{x-2}-\binom{x+1}{x}=4.$$
3. Add the chosen line of the Pascal's triangle.
4. How many subsets can be created from the set of n elements?
5. Prove assertion:
$$\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+2^3\binom{n}{3}+\cdots+2^n\binom{n}{n}=3^n.$$
6. There are 5 balls in the urn of the same shape but of different colour. How many possibilities are to successively draw all of them without their replacement and with depending on the order of chosen balls?
7. How many different numbers of five digits is possible to create from numbers 0, 1, 4, 7, 9 (without replacement of digits). How many of them are even?
8. How many different permutations are possible to create from the letters of the word MISSISSIPPI?
9. What is the number of possibilities of seating 5 women and 5 men around the round table under the condition that only different gender must be next to each other.
10. The group consists of 36 members. What is the number of possibilities of electing chairperson, vice chairperson, secretary, treasurer (under the condition that each member of the group can hold only one post).
11. There are 12 products in the set and 3 of them are defective. What is the number of possibilities of choosing
    a) 6 products,
    b) 6 flawless products,
    c) 6 products, where just 2 are defective.
12. There are 6 white and 5 red balls in the urn. How many possibilities are there to draw 4 balls on condition that at least two of them must be white?

13. There are 12 different kinds of postcards in the shop. How many possibilities are there to buy
    a) 15 postcards,
    b) 7 different postcards?
14. What is the sum of all numbers of four digits created from numbers 1, 3, 5, 7 (without replacement of digits).
15. How many possibilities are there to place 7 white and 2 black balls into 9 boxes (it is not necessary that each box must contain some ball)?
16. How many possibilities are there to distribute 10 blue, 15 red and 8 green balls among 4 children (each child must receive at least one ball from all 3 types).

Solution.
1. $\binom{9}{5}$;
2. 5;
3. 2^n;
4. 2^n;
6. $P(5) = 5! = 120$;
7. 96; 42;
8. $P'_{4,4,2,1}(11) = 34650$;
9. $2 \cdot 5! \cdot 5!$;
10. $V_4(36) = 1413720$;
11. a) $C_6(12) = 924$; b) $C_6(9) = 84$; c) $C_2(3) \cdot C_4(9) = 378$;
12. 265;
13. a) $C'_{15}(12) = 7726160$; b) $C_7(12) = 792$;
14. 106656;
15. $C'_7(9) \cdot C'_2(9) = 289575$;
16. 1070160.

1.2 Random Experiment and Random Event

1.2.1 Exercises 1.2

1. We throw a fair die until the outcome is number 6.
   a) Determine sample space Ω.
   b) Describe all possible favourable outcomes regarding event: „random experiment finishes on the second roll“.
   c) How many possible favourable outcomes regarding event: „random experiment finishes on the third roll “ are there?
2. Consider an industrial filter which undergoes three different laboratory tests. Event A_i indicates that the filter stands the i-th test, $i = 1, 2, 3$. Describe the event that the filter stands

   a) only the 1-th test,
   b) only the 1-th and 2-th test,
   c) all tests,
   d) at least one test,
   e) at least two tests,
   f) exactly one test,
   g) exactly two tests,
   h) at maximum one test.

3. Determine sample space Ω according to
   a) number of defective products from 50 checked products,
   b) number of vehicles which are fueled in a petrol station during one day,
   c) time of withdrawal from ATM during one order

4. Simplify following expressions:
   a) $(A \cup B) \cap \left(A \cup \overline{B}\right)$,
   b) $\left(\overline{A \cup B}\right) \cup \left(\overline{A \cup \overline{B}}\right)$,
   c) $\left(\overline{A} \cap B\right) \cup \left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap \overline{B}\right)$.

5. Event A indicates that randomly selected natural number is divisible by number 5, similarly event B indicates that the last digit of this number is 0. Determine the meaning of following expressions:
   a) $A \cap B$,
   b) $A \cup B$,
   c) $\overline{A} \cap B$,
   d) $A \cup \overline{B}$,
   e) $\overline{A \cap \overline{B}}$.

6. The mechanism of boiler room consists of an engine part and two boilers. Event A and B_1, B_2 indicates that engine part and the first and the second boiler work properly. Using these events express event C and $\overline{C}$, where event C means that boiler room is capable of working if
   a) the engine part and at least one boiler works properly,
   b) the engine part and the first boiler works properly.

Solution.

1. a) $\Omega = \{[6], [1, 6], [2, 6], [3, 6], \dots, [5, 6], [1, 1, 6], [1, 2, 6], [1, 3, 6], \dots, [1, 5, 6], [2, 1, 6], [2, 2, 6], \dots, [5, 5, 6], [1, 1, 1, 6], \dots\}$; b) [1, 6], [2, 6], [3, 6], [4, 6], [5, 6]; c) $[x, y, 6]$, where $x, y \in \{1, 2, \dots, 5\}$, total 5^2 possibilities;
2. a) $A_1 \cap \overline{A_2} \cap \overline{A_3}$; b) $A_1 \cap A_2 \cap \overline{A_3}$; c) $A_1 \cap A_2 \cap A_3$; d) $A_1 \cup A_2 \cup A_3$;
   e) $\left(A_1 \cap A_2 \cap \overline{A_3}\right) \cup \left(A_1 \cap \overline{A_2} \cap A_3\right) \cup \left(\overline{A_1} \cap A_2 \cap A_3\right) \cup (A_1 \cap A_2 \cap A_3)$;
   f) $\left(A_1 \cap \overline{A_2} \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap A_2 \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap \overline{A_2} \cap A_3\right)$;
   g) $\left(A_1 \cap A_2 \cap \overline{A_3}\right) \cup \left(A_1 \cap \overline{A_2} \cap A_3\right) \cup \left(\overline{A_1} \cap A_2 \cap A_3\right)$;
   h) $\left(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\right) \cup \left(A_1 \cap \overline{A_2} \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap A_2 \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap \overline{A_2} \cap A_3\right)$;
3. a) $\Omega = \{0, 1, 2, \dots, 50\}$; b) $\Omega = \{0, 1, 2, \dots\}$; c) $\Omega = \{x;\ x \in \mathbb{R}^+\}$;

4. a) A; b) $\overline{A}$; c) $\overline{A} \cup \overline{B}$;
5. a) last digit is 0; b) divisibility by 5; c) impossible event;
   d) sure event; e) all numbers except numbers with last digit 5;
6. a) $C = A \cap (B_1 \cup B_2)$; $\overline{C} = \overline{A} \cup (\overline{B_1} \cap \overline{B_2})$; b) $C = A \cap B_1$, $\overline{C} = \overline{A} \cup \overline{B_1}$.

1.3 Probability of Random Event

1.4 Classical Definition of Probability

1.4.1 Exercises 1.4

1. A random experiment lies in rolling a six-sided die. An event A occurs, if outcome is even number and an event B occurs, if the outcome is a number greater than or equal to 4. Compute probabilities $P(A)$, $P(B)$, $P(\overline{A})$, $P(\overline{B})$, $P(A \cup B)$, $P(A \cap B)$, $P(A - B)$, $P(B - A)$.
2. A random experiment lies in rolling blue and red six-sided dies. The outcome is that the number on the red die is greater than the number of blue die. What is the probability that the number on red die is 5?
3. There are 10 000 tickets in the lottery, where 100 of them win 1000 \$, 100 win 500 \$ and 1000 win 10 \$. What is the probability that one randomly selected ticket
   a) win,
   b) win 500 \$,
   c) win most highly 500 \$,
   d) win at least 500 \$.
4. Suppose a randomly chosen 4 cards from the pack of 32 cards. Compute the probability that at least one of these cards is the ace.
5. What is the probability that for throwing with 6 fair dies is (are)
   a) different numbers on each die,
   b) number 6 on all dies,
   c) exactly 5 dies,
   d) exactly 4 dies,
   e) at least 4 dies,
   f) only even numbers,
   g) all numbers same.
6. A random experiment lies in throwing with three fair dies. Which of two following events has greater probability? An event A indicates that sum of numbers is 11 and an event B indicates that sum is 12.
7. Calculate the probability that it is possible to create a triangle from three given abscissae which are randomly chosen from

   a) 4 abscissae in length 4, 6, 8 a 10,
   b) 5 abscissae in length 5, 8, 10, 13 a 15.

8. Consider you know only 25 questions from total number 50. If you randomly draw 3 questions compute the probability that you know
   a) all 3 qestions,
   b) exactly 2 questions?
9. Around a military area is 9 watchtowers and only 6 of them are guarded. The enemy randomly shells 3 watchtowers. What is the probability that enemy shells
   a) 3 guarded watchtowers,
   b) 2 guarded and 1 unguarded watchtowers,
   c) at least 1 unguarded watchtower?
10. Among 100 screws are 5 rejects. What is the probability that in set of 10 randomly chosen screws
    a) is exactly one reject,
    b) are most highly 2 rejects?
11. In an urn are 6 red, 3 blue and 3 white balls. Randomly choose 4 of them and compute the probability that
    a) all 4 balls are red,
    b) 3 balls are red and 1 is blue,
    c) 2 balls are red, 1 blue a 1 white.
12. In a delivery of 50 wall clocks is 46 in working order. Due to a control ware chosen 4 pieces. What is the probability that
    a) all clocks are functional,
    b) most highly 3 clocks are functional,
    c) 2 clocks are functional and 2 functionless,
    d) all clocks are functionless?
13. 10 cars are randomly parked in one line. What is the probability that 3 given cars are parked beside themselves.
14. The probability that two certain soldiers from same troop are chosen for quaternary guards is 1/20. How many members has described troop?

1. $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$; $P(\overline{A}) = \frac{1}{2}$; $P(\overline{B}) = \frac{2}{3}$; $P(A \cup B) = \frac{2}{3}$; $P(A \cap B) = \frac{1}{6}$; $P(A - B) = \frac{1}{3}$; $P(B - A) = \frac{1}{6}$;
2. $\frac{4}{15}$;
3. a) 0.12; b) 0.01; c) 0.11; d) 0.02;
4. 0.431;
5. a) 0.0154; b) $2.143 \cdot 10^{-5}$; c) $6.43 \cdot 10^{-4}$; d) $8.038 \cdot 10^{-3}$; e) 0.0087; f) 0.0156; g) $1.286 \cdot 10^{-4}$;
6. $P(A) = 0.125$; $P(B) = 0.116$;
7. a) $\frac{3}{4}$; b) $\frac{7}{10}$;
8. a) 0.117; b) 0.383;

9. a) 0.238; b) 0.536; c) 0.762;
10. a) 0.339; b) 0.993;
11. a) 0.030; b) 0.121; c) 0.273;
12. a) 0.709; b) 0.291; c) 0.027; d) $4.34 \cdot 10^{-6}$;
13. $\frac{1}{15}$;
14. 16.

1.5 Geometrical Definition of Probability

1.5.1 Exercises 1.5

1. In the military area is stretched a telephonic cable between a commanding station and a bridge, length is 600 m. Somewhere (randomly chosen point B) is the cable interrupted. Compute the probability that K has the distance from commanding station
   a) greater than 75 m,
   b) most highly 10 m?
2. There are 4 concentric circles with radii 2, 3, 4 and 5. In the circle with radius 5 is randomly chosen point K. What is the probability that K belongs to
   a) the internal circle,
   b) the circle with radius 3,
   c) the middle annulus?
3. A randomly chosen traveller, who can equipollently take both links (designate A and B), comes to a bus stop. Compute the probability that
   a) bus A will come earlier than bus B,
   b) bus A or B will come in 5 minutes.
4. There are two steamers and both of them can reach the port just once in one day, anytime and independently of them. The first steamer will stay in the port for one hour and the second steamer for two hours. What is the probability that one steamer will have to wait for the other (in the port is not enough place for both steamers at the same time)?
5. What is the probability that sum of two randomly chosen positive numbers is less than 1 and that the product of these numbers is greater than $\frac{2}{9}$.
6. An abscissa, which has tne length 200 mm, is randomly divided into 3 parts. Compute the probability that the middle part is less than or equal to 10 mm.

Solution.

1. a) 0.875; b) 0.167;
2. a) 0.16; b) 0.36; c) 0.28;
3. a) 5/8; b) 1/2;

4. 0.121;
5. 0.013;
6. 0.0975.

1.6 Conditional Probability

1.7 Multiplication Law for Probability

1.7.1 Exercises 1.6 a 1.7

1. In a box is 21 wraped glasses, 15 of them have some decoration and the rest is without decoration. Randomly choose several glasses. What is the probability that second chosen glass has decoration given that the first one has decoration too. Solve the problem for drawing
   a) without replacement,
   b) with replacement.
2. The task is to shoot at two masked targets, shooting at second one is allowed only in case that the first shot is successful. The probability that the rifleman hit the first target is $\frac{3}{5}$, the probability that the rifleman hit both targets is $\frac{2}{5}$. What is the probability that the second target will be hit.
3. Worker A produces 60 products per day but 10 % of them are spoilages. Similarly worker B produces 40 products per day, the percentage of spoilages is 5 %. What is the probability that randomly chosen product is spoilage and comes from production of
   a) worker A,
   b) worker B?
4. Draw without replacement two balls from the urn where are a white a b black balls. What is the probability that the second ball is white given that the first ball is white?
5. Draw without replacement three products from 5 products where are 3 rejects. Consider events A_1: „the first chosen product is quality", A_2: „the second chosen product is reject", A_3t: „ the third chosen product is reject". Compute the probability of the intersection of events A_1, A_2, A_3.
6. Draw without replacement 11 cards from pack of cards (32 cards). What is the probability that an ace is chosen exactly in the last drawing.
7. There are 4 black and 4 white balls in the urn. Draw without replacement four times two balls. What is the probability that in all drawings are 1 black and 1 white ball.
8. There are $2n$ seats in the row of a cinema. What is the probability that there are no neighbours of same sex given that this row is occupied by n men and n women.

9. Roll two fair dies. Consider events A, B and C: „the sum is divisible by 2, 3 and 4“. Verify pair independences and independence of events A, B, C.

10. Roll two fair dies. Compute the probability of given events and verify if defined events are independent.
    a) the number on the second die is greater than 2 given that the number on the first die is 2,
    b) the sum of numbers is greater than 6 given that the number on the first die is 2,
    c) the number on the second die is less than 4 given that the number on the first die is odd,
    d) the sum of numbers is greater than 9 given that the number on the first die is even.

11. There are two urns and both of them contain one white and one black ball. Choose one ball from each urn. Consider events A: „the ball chosen from the first urn is white“, B: „the ball chosen from the second urn is black“, C: „balls have same colours“. Verify pair independences and independence of events A, B, C.

Solution.
1. a) $\frac{7}{10}$; b) $\frac{5}{7}$;
2. $\frac{2}{3}$;
3. a) 0.06; b) 0.02;
4. $\frac{a-1}{a+b-1}$;
5. 0.2;
6. 0.0370;
7. 0.229;
8. $2 \cdot \frac{(n!)^2}{(2n)!}$;
9. A, B independent; A, C a B, C dependent; A, B, C dependent;
10. a) 2/3; yes b) 1/3; no c) 1/2; yes d) 2/9; no;
11. pair independence yes; independence no.

1.8 Addition Law for Probability

1.8.1 Exercises 1.8

1. There are 6 balls with numbers 1, 2, . . . , 6 in the urn. Consider drawings without replacement. What is the probability that at least in one case is the number of ball and the number of draw same?

2. Roll three times a fair die. What is the probability that it is at least in one case number 1.

3. The probability of an incremental investment is 0.3. Compute the probability that from 6 independent investments is at least one incremental?
4. A seed germinates with probability $\frac{2}{3}$. Consider 6 planted seeds, what is the probability that at least one seed germinates?
5. A bulb shine, if a rifleman hits a target. 4 competitors shoot independently and hit the target with probabilities 0.55, 0.42, 0.36 and 0.22. Each rifleman shoots just once. What is the probability that the bulb
   a) shines,
   b) does not shine?
6. There are 8 independent machines in a workroom. Consider i-th, $i = 1, 2, \dots, 8$, machine does not require some repair with probabilities 0.80, 0.89, 0.84, 0.90, 0.85, 0.92, 0.86 and 0.95. What is the probability that
   a) none of machines requires the repair,
   b) at least one machine requires the repair,
   c) only machines 1, 3 and 5 require the repair?

Solution.
1. 0.632;
2. $1 - \left(\frac{5}{6}\right)^3$;
3. 0.882;
4. $\frac{728}{729}$;
5. a) 0.870; b) 0.130;
6. a) 0.344; b) 0.656; c) $2.89 \cdot 10^{-3}$.

1.9 Law of Total Probability and Bayes' Formula

1.9.1 Exercises 1.9

1. There are 5 black and 15 white balls in the urn. Choose one ball, give it back and add 20 extra balls with exactly same colour as the first chosen ball. What is the probability that next chosen ball is black?
2. There are 23 students in a study group. 8 of them pass an exam with probability 0.9, 12 students with probability 0.6 and 3 students with probability 0.4. Compute the probability that randomly chosen student passes the exam?
3. A PC market is supplied with two providers. The first covers 80 % of demand, where 75 % of it are PC with processor Intel. The second covers 20 %, where 60 % of it are PC with processor Intel. What is the probability that randomly chosen computer
   a) is equiped with processor Intel,
   b) with processor Intel is from the first (second) provider?

4. An insurance company differentiates three kinds of drivers, designate A, B and C. The driver A meets with an accident with probability 0.03, driver B 0.06 and driver C 0.10. According to experience, 70 % of insurance contracts are with drivers A, 20 % B and 10 % C. If an insured driver meets with an accident, what is the probability that it is driver
   a) A,
   b) B,
   c) C?
5. There are 25 soldiers in a troop, where 5 of them are excellent shooters, 11 good, 7 average and 2 bad. Probabilities of hit are step by step 0.9, 0.7, 0.5 and 0.3.
   a) What is the probability that randomly chosen soldier hits the target?
   b) Randomly chosen soldier does not hit the target. What group does he most likely belong to?

Solution.
1. 0.25;
2. 0.678;
3. a) 0.72; b) 0.833 resp. 0.167;
4. a) 0.488; b) 0.279; c) 0.233;
5. a) 0.652; b) average.