Conditional Probability and Independence

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Conditional Probability

Definition

The conditional probability $P(A/B)$ – the probability of A if we know that an event B occurs $(P(B) > 0)$ is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

$P(A/B)$– the conditional probability of A given B

Multiplication law for probability

Using the formula of the conditional probability we can write

$$P(A \cap B) = P(A/B) \cdot P(B) = P(B/A) \cdot P(A).$$

Let us assume we have s events, then we get

$$P(A_1 \cap A_2 \cap \cdots \cap A_s) = P(A_1) \cdot P(A_2/A_1) \cdots P(A_s/A_1 \cap A_2 \cap \cdots \cap A_{s-1}).$$

For events A, B, C:

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B).$$

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For events A, B, C:

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B).$$

Independent Events

If $P(A/B) = P(A)$ then we say that an event A is **independent** on an event B. Independence of two events is mutual. If A is independent on B, then also B is independent on A, which means $P(B/A) = P(B)$.

If A and B are independent then

$$P(A \cap B) = P(A) \cdot P(B).$$

Given formula is a necessary and sufficient condition of independence.

Addition Law for Probability

The probability of the union of events A and B is equal to

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For s events $A_1, A_2, \dots, A_s$ we can write

$$P(A_1 \cup A_2 \cup \cdots \cup A_s) = \sum_{i=1}^{s} P(A_i) - \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} P(A_i \cap A_j) + \\ + \sum_{i=1}^{s-2} \sum_{j=i+1}^{s-1} \sum_{k=j+1}^{s} P(A_i \cap A_j \cap A_k) + \cdots + \\ + (-1)^{s-1} P\left(\bigcap_{i=1}^{s} A_i \right).$$

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Addition Law for Probability

For 3 events A, B and C we get the formula

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) - \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) + \\ & + P(A \cap B \cap C). \end{aligned}$$

Addition Law for Probability

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

It is possible to generalize given formula for more than two events. Let us assume $A_1, A_2, \ldots, A_s$ are mutually exclusive events, then

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Example 1

Three persons A, B a C will vote independently about certain agreement. The probability that the person A will not vote in the favour of the agreement is 0.7, the person B will not vote aye with the probability 0.5 and the person C with probability 0.3. The agreement will be rejected if at least one person countervotes (will not agree).
What is the probability of rejection?

Example 1

$$\begin{aligned}P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) - \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) + \\ & + P(A \cap B \cap C) = \\ = {} & 0.7 + 0.5 + 0.3 - \\ & - 0.7 \cdot 0.5 - 0.7 \cdot 0.3 - 0{,}5 \cdot 0.3 + \\ & + 0.7 \cdot 0.5 \cdot 0.3 = 0.895\end{aligned}$$

or

$$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C}) =$$

$$= 1 - P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C}) = 1 - 0.3 \cdot 0.5 \cdot 0.7 = 0.895$$

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Example 2

We roll two dice. What is the probability that we get two numbers five if we know that the sum of given numbers is divisible by five?

- the event A ... we get two 5
- the event B ... the sum is divisible by five
  (1+4, 4+1, 2+3, 3+2, 4+6, 6+4, 5+5)

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Example 2

- $$P(A \cap B) = \frac{1}{36}$$
- $$P(B) = \frac{7}{36}$$
- $$P(A/B) = \frac{\frac{1}{36}}{\frac{7}{36}} = \frac{1}{7}$$

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Example 3

We have 32 playing cards and we choose randomly 1 card. Let us denote the event A – the chosen card is green,
the event B – the chosen card is an ace.
Calculate the probability of events A, B, $A \cap B$, $A \cup B$. Are the events A and B independent?

- $$P(A) = \frac{8}{32} = \frac{1}{4}$$
- $$P(B) = \frac{4}{32} = \frac{1}{8}$$

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Example 3

- $$P(A \cap B) = \frac{1}{32} = P(A)P(B/A) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$
- $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{8} - \frac{1}{32} = \frac{11}{32}$$
- $$P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32} = P(A \cap B)$$

  The events A and B are independent.

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  The events A and B are independent.

Example 4

The first worker produces daily 60 products, 10% of them are defective products. The second one produces 40 products, 5% of them are defective products. What is the probability that randomly chosen product is defective and was made by the first (or the second) worker?

Example 4

- the event A_1 …the product was made by the 1st worker

$$P(A_1) = \frac{60}{100} = \frac{6}{10} = 0.6$$

- the event A_2 …the product was made by the 2nd worker

$$P(A_2) = \frac{40}{100} = \frac{4}{10} = 0.4$$

- the event B …the product is defective

$$P(B) = \frac{6+2}{100} = \frac{8}{100} = 0.08$$

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Example 4

- The product is defective and was made by the 1st worker

$$P(A_1 \cap B) = ?$$

- The product is defective if we know that it was made by the 1st worker

$$P(B/A_1) = 0.1$$

-

$$P(A_1 \cap B) = P(A_1) \cdot P(B/A_1) = 0.6 \cdot 0.1 = 0.06$$

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Example 4

- The product is defective and was made by the 2nd worker

$$P(A_2 \cap B) = ?$$

- The product is defective if we know that it was made by the 2nd worker

$$P(B/A_2) = 0.05$$

- $$P(A_2 \cap B) = P(A_2) \cdot P(B/A_2) = 0{,}4 \cdot 0.05 = 0.02$$

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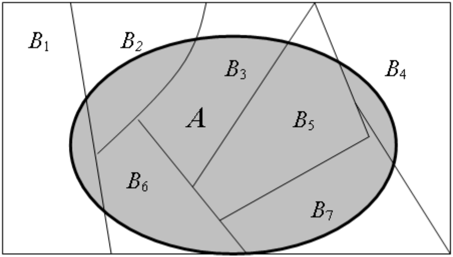
Law of Total Probability

Let us assume that we have mutually exclusive events $B_1, B_2, \ldots, B_n$, $(B_i \cap B_j = \emptyset, i \neq j)$ which fulfil

$$\Omega = \bigcup_{i=1}^{n} B_i \Rightarrow P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) = 1$$

and the conditional probability $P(A/B_i)$ $i = 1, 2, \ldots, n$ are known. We would like to calculate the probability of the event A.

Law of Total Probability



The probability of the event A is

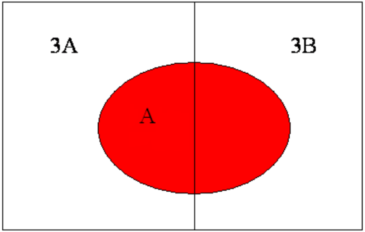
$$P(A) = P(B_1) \cdot P(A/B_1) + \cdots + P(B_n) \cdot P(A/B_n) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

Example

There are 13 boys and 16 girls in the class 3.A and 14 boys and 12 girls in the class 3.B. We choose 1 student from each class by random and between these 2 students we randomly select one. What is the probability that the selected student is a boy?

Example

- the event A . . . the chosen student is a boy
- the event B_1 . . . the chosen student is from 3.A
- the event B_2 . . . the chosen student is from 3.B



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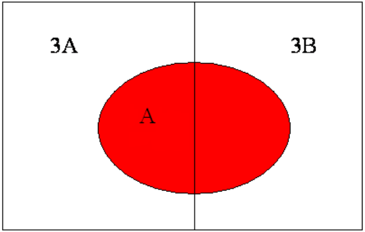

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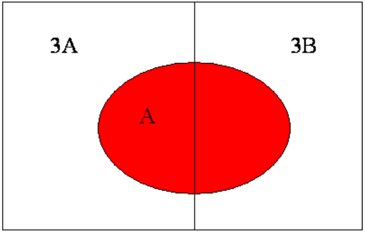
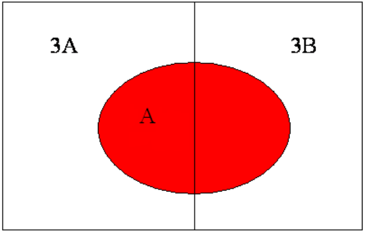


Example

- $P(B_1) = P(B_2) = \frac{1}{2}$
- $P(A/B_1) = \frac{13}{29}$
- $P(A/B_2) = \frac{14}{26}$



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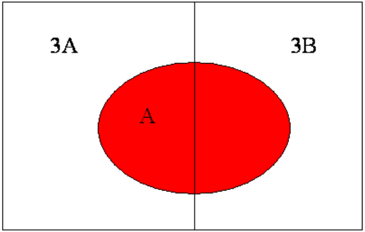

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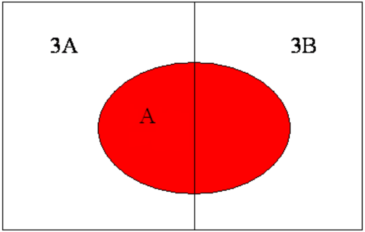


Example

$$P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) = \frac{1}{2} \cdot \frac{13}{29} + \frac{1}{2} \cdot \frac{14}{26} = 0.493$$

Bayes' Formula

The conditional probabilities $P(B_i/A)$ can be obtained using **Bayes' Formula**, which we can derive from the multiplication law for probability and the law of total probability:

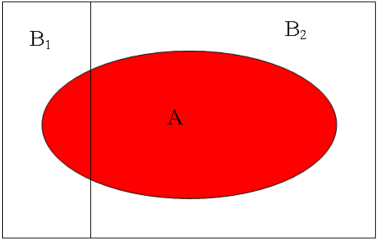
$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{j=1}^{n} P(B_j) \cdot P(A/B_j)}, i = 1, 2, \ldots, n.$$

Example

The polygraph is an instrument used to detect physiological signs of deceptive behaviour. Although it is often pointed out that the polygraph is not a lie detector, this is probably the way most of us think of it.
Let us assume that the polygraph test is indeed very accurate and that it decide "lie" or truth correctly with probability 0.95. Now we consider a randomly chosen individual who takes the test and is determined to be lying. What is the probability that this person did indeed lie?
Suppose that we are dealing with a large hones population, let us say that one of a thousand would tell a lie in the given situation.

Example

- the event A . . . the polygraph reading says the person is lying
- the event B_1 . . . the person tells a lie
- the event B_2 . . . the person tells a truth



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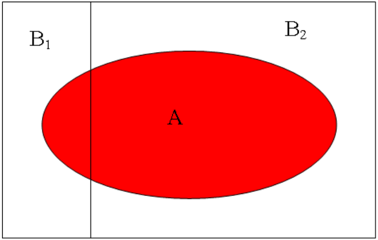

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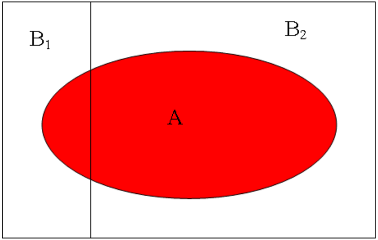

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- $P(B_1) = 0.001$
- $P(B_2) = 0.999$
- $P(A/B_1) = 0.95$
- $P(A/B_2) = 0.05$

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Example

The probability that the person is lying if we know that the polygraph determined him as a liar is

$$P(B_1/A) = \frac{P(B_1) \cdot P(A/B_1)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)} = \\ = \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} \doteq 0.02.$$