Random Variables

Jiří Neubauer

Department of Econometrics FVL UO Brno
office 69a, tel. 973 442029
email:Jiri.Neubauer@unob.cz

Random Variable

Many random experiments have numerical outcomes.

Definition

A **random variable** is a real-valued function $X(\omega)$ defined on the sample space Ω.

The set of possible values of the random variable X is called the **range** of X.

$$M = \{x; X(\omega) = x\}.$$

Random Variable

- We denote random variables by capital letters $X, Y, \ldots$ (eventually $X_1, X_2, \ldots$) and their particular values by small letters $x, y, \ldots$. Using random variables we can describe random events, for example $X = x, X \leq x, x_1 < X < x_2$ etc.

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- Examples of random variables:
  - the number of dots when a die is rolled, the range is $M = \{1, 2, \ldots 6\}$
  - the number of rolls of a die until the first 6 appears, the range is $M = \{1, 2, \vdots\}$
  - the lifetime of the lightbulb, the range is $M = \{x; x \geq 0\}$,

Random Variable

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Random Variable

According to the range M we separate random variables to

- **discrete** . . . M is finite or countable,
- **continuous** . . . M is a closed or open interval.

Random Variable

Examples of discrete random variables:

- the number of cars sold at a dealership during a given month, $M = \{0, 1, 2, \ldots\}$
- the number of houses in a certain block, $M = \{1, 2, \ldots\}$
- the number of fish caught on a fishing trip, $M = \{0, 1, 2, \ldots\}$
- the number of heads obtained in three tosses of a coin, $M = \{0, 1, 2, 3\}$

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Examples of continuous random variables:

- the height of a person, $M = (0, \infty)$
- the time taken to complete an examination, $M = (0, \infty)$
- the amount of milk in a bottle, $M = (0, \infty)$

Random Variable

For the description of random variables we will use some functions:

- a cumulative distribution function $F(x)$,
- a probability function $p(x)$ – only for discrete random variables,
- a probability density function $f(x)$ – only for continuous random variables.

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- a cumulative distribution function $F(x)$,
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- a probability density function $f(x)$ – only for continuous random variables.

and some measures:

- measures of location,
- measures of dispersion,
- measures of concentration.

Cumulative Distribution Function

Definition

Let X be any random variable. The **cumulative distribution function** $F(x)$ of the random variable X is defined as

$$F(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

Cumulative Distribution Function

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- for every real x: $0 \leq F(x) \leq 1$,

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$$\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1,$$
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  if range of X is $M = \{x; x \in (a, b\rangle\}$ then $F(a) = 0$ a $F(b) = 1$,
- for every real numbers x_1 and x_2: $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$.

Discrete Random Variable

For a discrete random variable X, we are interested in computing probabilities of the type $P(X = x_k)$ for various values x_k in range of X.

Definition

Let X be a discrete random variable with range $\{x_1, x_2, \dots\}$ (finite or countably infinite). The function

$$p(x) = P(X = x)$$

is called the **probability function** of X.

Note: probability function = probability mass function

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- for every real number x, $0 \leq p(x) \leq 1$,
- $$\sum_{x \in M} p(x) = 1$$
- for every two real numbers x_k and x_l $(x_k \leq x_l)$:

$$P(x_k \leq X \leq x_l) = \sum_{x_i = x_k}^{x_l} p(x_i).$$

Probability Function

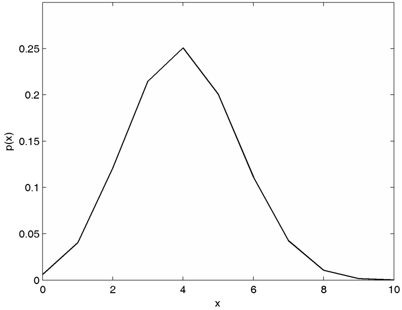
The probability function $p(x)$ can be described by

- the table,

X	x_1	x_2	$\dots$	x_i	$\dots$	$\sum$
$p(x)$	$p(x_1)$	$p(x_2)$	$\dots$	$p(x_i)$	$\dots$	1

Probability Function

- the graph $[x, p(x)]$,



Probability Function

- the formula, for example

$$p(x) = \begin{cases} \pi(1-\pi)^x & x = 0, 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$

where π is a given probability.

Example

The shooter has 3 bullets and shoots at the target until the first hit or until the last bullet. The probability that the shooter hits the target after one shot is 0.6. The random variable X is the number of the fired bullets. Find the probability and the cumulative distribution function of the given random variable. What is the probability that the number of the fired bullets will not be larger then 2?

Example

Random variable X is discrete with the range $M = \{1, 2, 3\}$. The probability function is:

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- $p(1) = P(X = 1) = 0.6$,
- $p(2) = P(X = 2) = 0.4 \cdot 0.6 = 0.24$,
- $p(3) = P(X = 3) = 0.4 \cdot 0.4 \cdot 0.6 + 0.4 \cdot 0.4 \cdot 0.4 = 0.4 \cdot 0.4 = 0.16$.

Example

All results are summarized in the table

x	1	2	3	$\sum$
$p(x)$	0.6	0.24	0.16	1

The probability function can be described by the formula

$$p(x) = \begin{cases} 0.6 \cdot 0.4^{x-1} & x = 1, 2, \\ 0.4^2 & x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

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- $F(2) = P(X \leq 2) = p(1) + p(2) = 0.84,$
- $F(3) = P(X \leq 3) = p(1) + p(2) + p(3) = 1,$
- $F(4) = P(X \leq 4) = p(1) + p(2) + p(3) = 1.$

Example

We can write

$$F(x) = \begin{cases} 0 & x < 1, \\ 0.6 & 1 \leq x < 2, \\ 0.84 & 2 \leq x < 3, \\ 1 & x \geq 3. \end{cases}$$

Example

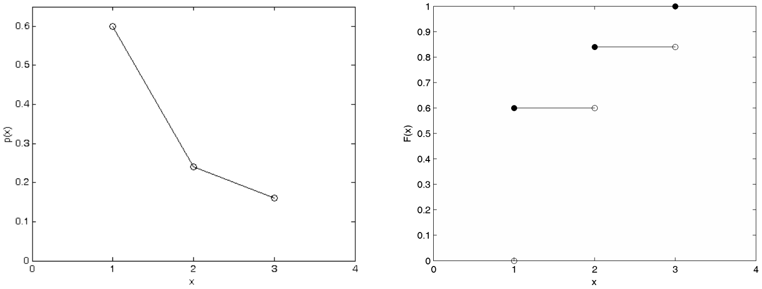

Figure: The probability and the cumulative distribution function

Example

What is the probability that the number of the fired bullets will not be larger then 2?

$$P(X \leq 2) = P(X = 1) + P(X = 2) = p(1) + p(2) = F(2) = \\ = 0.6 + 0.24 = 0.84.$$

Probability Density Function

If the cumulative distribution function is a continuous function, then X is said to be a continuous random variable.

Definition

The **probability density function** of the random variable X is a non-negative function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(t)dt, x \in \mathbb{R}.$$

Probability Density Function

Some properties of $f(x)$:

- $\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{M} f(x)dx = 1,$

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- $P(x_1 \leq X \leq x_2) = P(x_1 < X < x_2) = P(x_1 < X \leq x_2) =$
  $= P(x_1 \leq X < x_2) = F(x_2) - F(x_1) = \int\limits_{x_1}^{x_2} f(x)dx$

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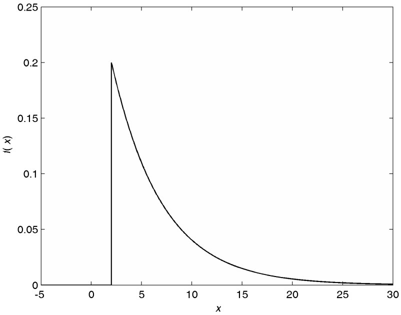
  $= P(x_1 \leq X < x_2) = F(x_2) - F(x_1) = \int\limits_{x_1}^{x_2} f(x)dx$

If X is a continuous random variable, then $P(X = x) = 0$.

Probability Density Function

The function $f(x)$ we can describe by a formula or a graph, for example

$$f(x) = \begin{cases} \frac{1}{5} e^{-\frac{x-2}{5}} & \text{pro } x > 2, \\ 0 & \text{pro } x \leq 2. \end{cases}$$



Example

The random variable X has the probability density function

$$f(x) = \begin{cases} cx^2(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine a constant c in order that $f(x)$ is a probability density function. Find a distribution function of the random variable X. Calculate the probability $P(0.2 < X < 0.8)$.

Example

The probability density function has to fulfil

$$\int_M f(x)dx = 1.$$

$$\int_0^1 cx^2(1-x)dx = c\int_0^1 (x^2 - x^3)dx = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 =$$
$$= c\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{c}{12} = 1,$$

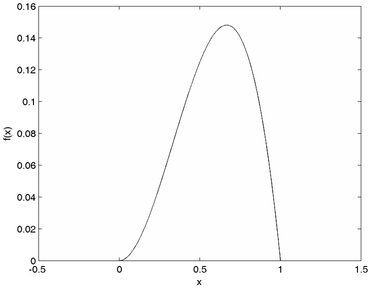
we get $c = 12$.

Example

The distribution function can be calculated by the definition of the probability density function. We can write for $0 < x < 1$

$$F(x) = \int_0^x 12t^2(1-t)dt = 12\int_0^x (t^2 - t^3)dt = 12\left[\frac{t^3}{3} - \frac{t^4}{4}\right]_0^x =$$
$$= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right] = 4x^3 - 3x^4.$$

$$F(x) = \begin{cases} 0 & x \leq 0, \\ x^3(4-3x) & 0 < x < 1, \\ 1 & \text{otherwise.} \end{cases}$$



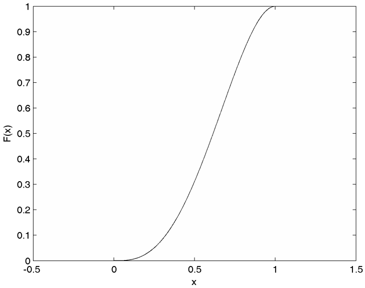


Figure: The probability density function and the cumulative distribution function

Example

Using the probability density function we can calculate

$$P(0.2 < X < 0.8) = \int_{0.2}^{0.8} 12x^2(1-x)dx = \left[4x^3 - 3x^4\right]_{0.2}^{0.8} = 0.792.$$

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If the distribution function is known, we can do simpler calculation

$$\begin{aligned} P(0.2 < X < 0.8) &= F(0.8) - F(0.2) = \\ &= 0.8^3(4 - 3 \cdot 0.8) - 0.2^3(4 - 3 \cdot 0.2) = 0.792. \end{aligned}$$

Example

A random variable X is described by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x \leq 0, \\ 1 - e^{-x} & x > 0. \end{cases}$$

Find a probability density function.

Example

Using mentioned formula

$$f(x) = \frac{dF(x)}{dx}$$

and the fact that $\frac{d}{dx}(1 - e^{-x}) = e^{-x}$ we get

$$f(x) = \begin{cases} 0 & x \leq 0, \\ e^{-x} & x > 0. \end{cases}$$

Measures of Location

The cumulative distribution function (the probability function or the probability density function) gives us the complete information about the random variable. Sometimes is useful to know some simpler and concentrated formulation of this information such as measures of location, dispersion and concentration.

The best known measures of location are a mean (an expected value), quantiles (a median, an upper and a lower quartile, . . .) and a mode.

Expected Value

Definition

The **mean (expected value)** $E(X)$ of the random variable X (sometimes denoted as μ) is the value that is expected to occur per repetition, if an experiment is repeated a large number of times. For the discrete random variable is defined as

$$E(X) = \sum_M x_i p(x_i),$$

for the continuous random variable as

$$E(X) = \int_M x f(x) dx$$

if the given sequence or integral absolutely converges.

Expected Value

We mention some properties of the mean:

- the mean of the constant c is equal to this constant

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$$E(c) = c,$$

- the mean of the product of the constant c and the random variable X is equal to the product of the given constant c and the mean of X

$$E(cX) = cE(X),$$

Expected Value

- the mean of the sum of random variables $X_1, X_2, \ldots, X_n$ is equal to the sum of the mean of the given random variables,

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n),$$

Expected Value

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$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n),$$

- if $X_1, X_2, \ldots, X_n$ are independent, then the mean of their product is equal to the product of their means

$$E(X_1 X_2 \ldots X_n) = E(X_1)E(X_2)\ldots E(X_n).$$

Random Variables and Independency

The random variables $X_1, X_2, \dots, X_n$ are independent if and only if for any numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ is

$$P(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) = \\ = P(X_1 \le x_1) \cdot P(X_2 \le x_2) \cdots P(X_n \le x_n).$$

Let us have the random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, whose components $X_1, X_2, \dots, X_n$ are the random variables.
$F(\mathbf{x}) = F(x_1, x_2, \dots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n)$ is the cumulative distribution function of the vector $\mathbf{X}$ and
$F(x_1), F(x_2), \dots, F(x_n)$ are the cumulative distribution functions of the random variables $X_1, X_2, \dots, X_n$. The random variables $X_1, X_2, \dots, X_n$ are independent if and only if

$$F(x_1, x_2, \dots, x_n) = F(x_1) \cdot F(x_2) \cdots F(x_n).$$

Random Variables and Independency

If $\mathbf{X}$ is the random vector whose components are the discrete random variables, the function $p(\mathbf{x}) = p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$ is the probability function of the vector $\mathbf{X}$, $p(x_1), p(x_2), \ldots, p(x_n)$ are the probability functions of $X_1, X_2, \ldots, X_n$, then:
$X_1, X_2, \ldots, X_n$ are independent if and only if

$$p(x_1, x_2, \ldots, x_n) = p(x_1) \cdot p(x_2) \cdots p(x_n).$$

If $\mathbf{X}$ is the random vector whose components are the continuous random variables, the function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ is the probability density function of the vector $\mathbf{X}$, $f(x_1), f(x_2), \ldots, f(x_n)$ are the probability density functions of $X_1, X_2, \ldots, X_n$, then: $X_1, X_2, \ldots, X_n$ are independent if and only if

$$f(x_1, x_2, \ldots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n).$$

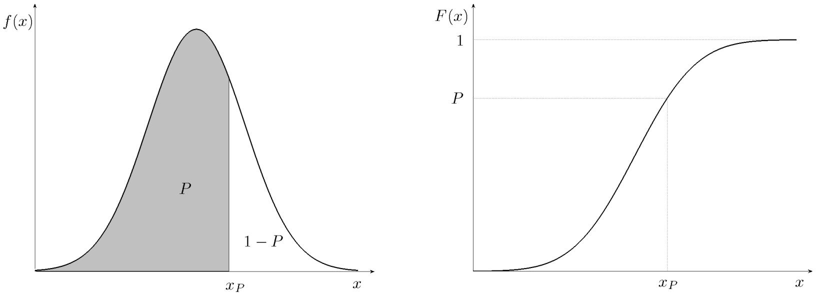
Quantile

Definition

100P% quantile x_P of the random variable with the increasing cumulative distribution function is such value of the random variable that

$$P(X \leq x_P) = F(x_P) = P, \quad 0 < P < 1.$$

Quantile



The quantile $x_{0.50}$ we call **median** $Me(X)$, it fulfils $P(X \leq Me(X)) = P(X \geq Me(X)) = 0.50$. The quantile $x_{0.25}$ is called the **lower quartile**, the quantile $x_{0.75}$ is called the **upper quartile**. The selected quantiles of some important distributions are tabulated.

Mode

Definition

The **mode** $Mo(X)$ is the value of the random variable with the highest probability (for the discrete random variable), or the value, where the function $f(x)$ has the maximum (for the continuous random variable).

Example

Find the mean (the expected value) and the mode of the random variable defined as the number of fired bullets (see the example before). The probability function is

$$p(x) = \begin{cases} 0.6 \cdot 0.4^{x-1} & x = 1, 2, \\ 0.4^2 & x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Example

The mean (the expected value) we get using the formula from the definition of $E(X)$

$$E(x) = \sum_{i=1}^{3} x_i p(x_i) = 1 \cdot 0.6 \cdot 0.4^0 + 2 \cdot 0.6 \cdot 0.4^1 + 3 \cdot 0.4^2 = 1.56.$$

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The mode is the value of the given random variable with the highest probability which is $Mo(X) = 1$, because $p(1) = 0.6$.

Example

The random variable X is described by the probability density function

$$f(x) = \begin{cases} 12x^2(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean (the expected value) and the mode.

Example

We calculate the mean using the definition formula

$$E(X) = \int_0^1 x f(x) dx = \int_0^1 x \cdot 12x^2(1-x) = 12 \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{3}{5} = 0.6.$$

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$$E(X) = \int_0^1 xf(x)dx = \int_0^1 x \cdot 12x^2(1-x) = 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{3}{5} = 0.6.$$

The mode is the maximum of the probability density function. We have to find the maximum of $f(x)$ on the interval $0 < x < 1$, $\frac{d}{dx}\left[12x^2(1-x)\right] = 12(2x - 3x^2) = 0$, $x(2-3x) = 0$, we get $x = 0$ or $x = 2/3$. The maximum of $f(x)$ is in $x = 2/3 \Rightarrow Mo(X) = 2/3$.

Example

Find the median,the upper and the lower quartile of the random variable X with the distribution function

$$F(x) = \begin{cases} 1 - \frac{1}{x^3} & x > 1, \\ 0 & x \leq 1. \end{cases}$$

Example

The quantile is defined by the formula $F(x_P) = P$.

$$1 - \frac{1}{x_P^3} = P,$$

$$x_P = \frac{1}{\sqrt[3]{1-P}}.$$

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$$x_P = \frac{1}{\sqrt[3]{1-P}}.$$

median $x_{0.50} = \frac{1}{\sqrt[3]{1-0.50}} = 1.260,$

lower quartile $x_{0.25} = \frac{1}{\sqrt[3]{1-0.25}} = 1.101,$

upper quartile $x_{0.75} = \frac{1}{\sqrt[3]{1-0.75}} = 1.587.$

Measures of Dispersion

The elementary and widely-used measures of dispersion are the variance and the standard deviation.

Variance

Definition

The **variance** $D(X)$ of the random variable X (sometimes denoted as σ^2) is defined by the formula

$$D(X) = E\left\{[X - E(X)]^2\right\}.$$

The variance of the discrete random variable is given by

$$D(X) = \sum_M [x_i - E(X)]^2 p(x_i),$$

the variance of the continuous random variable is

$$D(X) = \int_M [x - E(X)]^2 f(x) dx.$$

Variance

Some properties of the variance:

- $D(k) = 0$, where k is a constant,

Variance

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Variance

Some properties of the variance:

- $D(k) = 0$, where k is a constant,
- $D(kX) = k^2 D(X)$,
- $D(X + Y) = D(X) + D(Y)$, if X and Y are independent,
- $D(X) \geq 0$ for every random variable,

Variance

- $D(X) = E(X^2) - E(X)^2$,
  $$D(X) = E[X - E(X)]^2 = E[X^2 - 2XE(X) + E(X)^2] =$$
  $$= E(X^2) - E[2XE(X)] + E[E(X)^2] =$$
  $$= E(X^2) - 2E(X)E(X) + E(X)^2 = E(X^2) - E(X)^2$$
  For the discrete random variable
  $$D(X) = \sum_M x_i^2 p(x_i) - E(X)^2,$$
  for the continuous random variable
  $$D(X) = \int_M x^2 f(x)dx - E(X)^2.$$

Standard Deviation

Definition

The **standard deviation** $\sigma(X)$ of the random variable X is defined as the square root of the variance

$$\sigma(X) = \sqrt{D(X)}.$$

The standard deviation has the same unit as the random variable X.

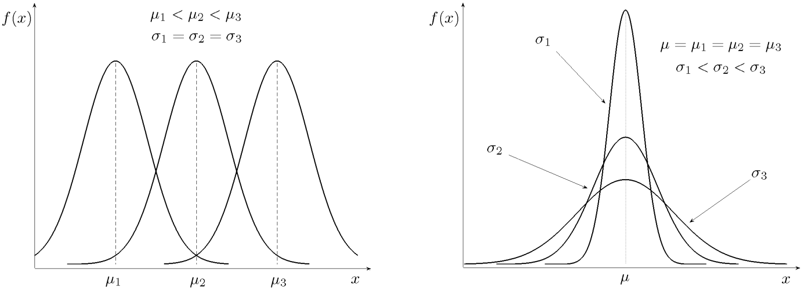

Figure: Relation between the mean and the standard deviation

Example

Find the variance and the standard deviation of the random variable defined as the number of fired bullets (see the example before).

Example

The mean is $E(X) = 1.56$ (see the previous example). For the purpose of calculation of the variance we use the formula

$$D(X) = E(X^2) - E(X)^2$$

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$$E(X^2) = \sum_M x_i^2 p(x_i) = \sum_{i=1}^{3} x_i^2 p(x_i) = 1^2 \cdot 0.6 + 2^2 \cdot 0.24 + 3^2 \cdot 0.16 = 3,$$

then

$$D(X) = 3 - 1.56^2 = 0.566.$$

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then

$$D(X) = 3 - 1.56^2 = 0.566.$$

The standard deviation is the square root of the variance

$$\sigma(X) = \sqrt{D(X)} = 0.753.$$

Example

Find the variance and the standard deviation of the random variable X with the probability density function

$$f(x) = \begin{cases} 12x^2(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example

The mean is $E(X) = 3/5$ (see the previous example).

$$D(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_M x^2 f(x)dx = \int_0^1 x^2 \cdot 12x^2(1-x)dx = 12\left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1$$
$$= \frac{2}{5} = 0.4,$$

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$$D(X) = \frac{2}{5} - \left(\frac{3}{5}\right)^2 = \frac{1}{25} = 0.04.$$

The standard deviation is

$$\sigma(X) = \sqrt{D(X)} = \frac{1}{5} = 0.2.$$

Measures of Concentration

We will focus on the measures describing the shape of random variables distribution (skewness and kurtosis). These measures are defined by moments.

Measures of Concentration

Definition

The r^{th} **moment** μ'_r of the random variable X is defined by the formula

$$\mu'_r(X) = E(X^r) \quad \text{for } r = 1, 2, \ldots .$$

The r^{th} moment of the discrete random variable is given by

$$\mu'_r(X) = \sum_M x_i^r p(x_i),$$

r^{th} moment of the continuous random variable is

$$\mu'_r(X) = \int_M x^r f(x) dx.$$

Measures of Concentration

Definition

The r^{th} **central moment** μ_r of the random variable X is defined by the formula

$$\mu_r(X) = E[X - E(X)]^r \quad \text{for } r = 2, 3, \ldots .$$

The r^{th} central moment of the discrete random variable is given by

$$\mu_r(X) = \sum_M [x_i - E(X)]^r p(x_i),$$

the r^{th} central moment of the continuous random variable is

$$\mu_r(X) = \int_M [x - E(X)]^r f(x) dx.$$

Skewness

Definition

The **skewness** $\alpha_3(X)$ is defined by the formula

$$\alpha_3(X) = \frac{\mu_3(X)}{\sigma(X)^3}.$$

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According to the values of the skewness we can tell whether distribution is symmetric or asymmetric.

- $\alpha_3 = 0$, distribution is symmetric,
- $\alpha_3 < 0$, distribution is skewed to the right,
- $\alpha_3 > 0$, distribution is skewed to the left.

Skewness

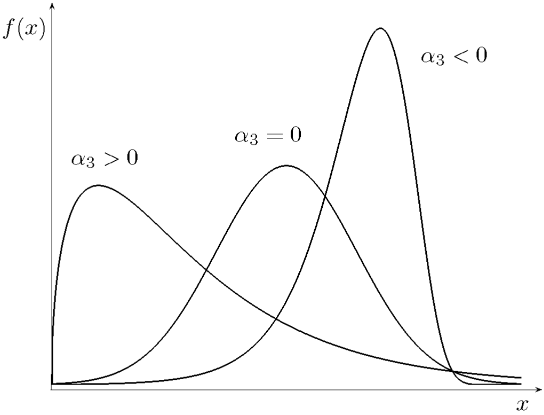

Figure: The skewness

Kurtosis

Definition

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Kurtosis is a measure of how outlier-prone a distribution is. The kurtosis of the normal distribution is 0. Distributions that are more outlier-prone than the normal distribution have kurtosis greater than 0; distributions that are less outlier-prone have kurtosis less than 0.

Kurtosis

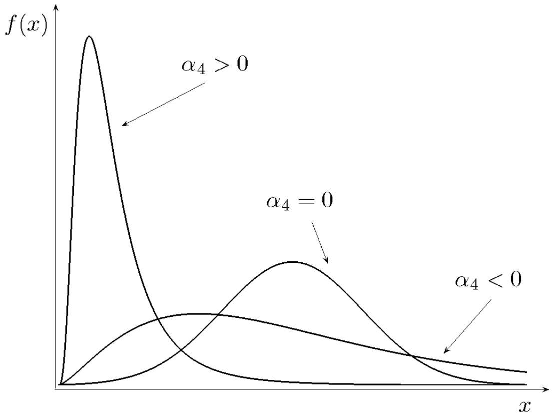

Figure: The kurtosis

Example

Calculate the skewness and the kurtosis of the random variable defined as the number of fired bullets (see the previous examples).

Example

First of all we have to calculate 3^{rd} and 4^{th} central moment. The mean of the given random variable is $E(X) = 1.56$, the standard deviation is $\sigma(X) = 0.753$.

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$$\mu_3 = \sum_{i=1}^{3}[x_i - E(X)]^3 p(x_i) = (1 - 1.56)^3 \cdot 0.6 + \\ +(2 - 1.56)^3 \cdot 0.24 + (3 - 1.56)^3 \cdot 0.16 = 0.393$$

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First of all we have to calculate 3^{rd} and 4^{th} central moment. The mean of the given random variable is $E(X) = 1.56$, the standard deviation is $\sigma(X) = 0.753$.

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$$\mu_4 = \sum_{i=1}^{3} [x_i - E(X)]^4 p(x_i) = (1 - 1.56)^4 \cdot 0.6 + \\ + (2 - 1.56)^4 \cdot 0.24 + (3 - 1.56)^4 \cdot 0.16 = 0.756$$

Example

The skewness is equal to

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = 0.922,$$

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the kurtosis is

$$\alpha_4 = \frac{\mu_4}{\sigma^4} - 3 = -0.644.$$

Example

Calculate the skewness and the kurtosis of the random X with the probability density function

$$f(x) = \begin{cases} 12x^2(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

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The mean of the given random variable is $E(X) = 3/5$. the standard deviation is $1/5$.

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$$\mu_3 = \int_0^1 [x - 0.6]^3 12x^2(1-x)dx = \cdots = -\frac{2}{875} = -0.00229,$$

Example

The mean of the given random variable is $E(X) = 3/5$. the standard deviation is $1/5$. First of all we calculate 3^{rd} and 4^{th} central moment.

$$\mu_3 = \int_0^1 [x - 0.6]^3 12x^2(1-x)dx = \cdots = -\frac{2}{875} = -0.00229,$$

$$\mu_4 = \int_0^1 [x - 0.6]^4 12x^2(1-x)dx = \cdots = \frac{33}{8750} = 0.00377.$$

Example

The skewness is equal to

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = -\frac{2}{7} = -0.286,$$

Example

The skewness is equal to

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = -\frac{2}{7} = -0.286,$$

the kurtosis is

$$\alpha_4 = \frac{\mu_4}{\sigma^4} - 3 = -\frac{9}{14} = -0.643.$$