Models of Continuous Random Variables

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The Uniform Distribution

Definition

If the probability density function of X is

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \alpha < x < \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, then X is said to have a **uniform distribution** on (α, β), written $X \sim R(\alpha, \beta)$

The Uniform Distribution

The cumulative distribution function can be calculated as

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{\alpha}^{x} \frac{1}{\beta - \alpha} dt = \cdots = \frac{x - \alpha}{\beta - \alpha} \quad \text{for} \quad \alpha < x < \beta.$$

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We obtain

$$F(x) = \begin{cases} 0 & x \le \alpha, \\ \dfrac{x - \alpha}{\beta - \alpha} & \alpha < x < \beta, \\ 1 & x \ge \beta. \end{cases}$$

The Uniform Distribution

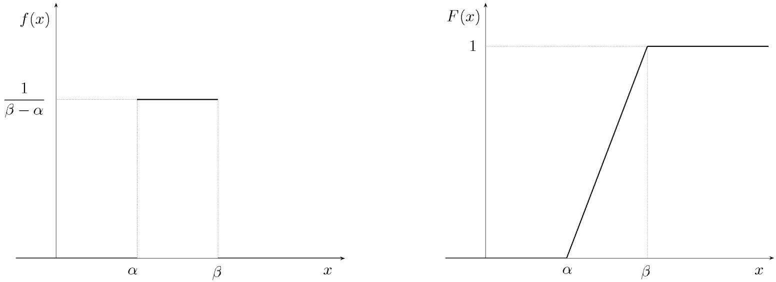

Figure: The probability density and the cumulative distribution function $R(\alpha, \beta)$

The Uniform Distribution

The table summarizes some basic information about the uniform distribution.

$E(X)$	$D(X)$	$\alpha_3(X)$	$\alpha_4(X)$	quantiles x_P	$Me(X)$
$\frac{\alpha+\beta}{2}$	$\frac{1}{12}(\beta-\alpha)^2$	0	-1.2	$\alpha + P(\beta-\alpha)$	$\frac{\alpha+\beta}{2}$

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Examples: a time we wait for a bus (buses go regularly every 10 minutes), a time we wait for a supply of bread in a grocery store (supplies are regular), calculation rounding mistakes, ...

The Uniform Distribution

Using:

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Example

Trams go regularly every 10 minutes. The passenger comes to the tram-stop at the arbitrary time. The random variable X is the time he/she has to wait for a tram.

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- Find the probability density function and the distribution function of X.
- What is the probability that the passenger will wait at most 3 minutes, at least 5 minutes, exactly 7 minutes.
- Calculate the mean, the median, the variance, the standard deviation and the 90 % quantile.

Example

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$$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 10, \\ 0 & \text{otherwise}, \end{cases}$$

the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x \leq 0, \\ \frac{x}{10} & 0 < x < 10, \\ 1 & x \geq 10. \end{cases}$$

Example

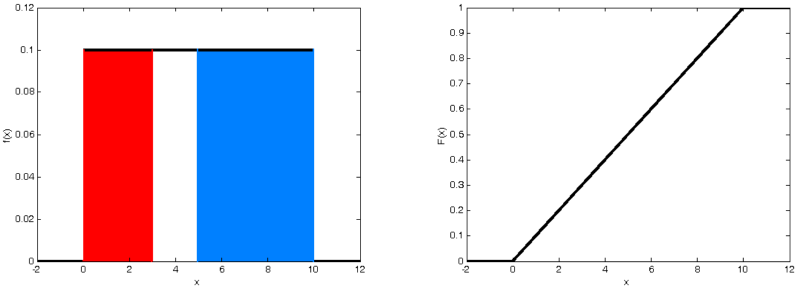

Figure: The probability density and the cumulative distribution function $R(0, 10)$

Example

The probability that the passenger will wait

- at most 3 minutes
  $P(X \leq 3) = \int\limits_0^3 \frac{1}{10} dx = \frac{1}{10}[x]_0^3 = 0.3$
  using the distribution function
  $P(X \leq 3) = F(3) = \frac{3}{10} = 0.3$

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- at least 5 minutes
  $$P(X \geq 5) = \int_5^{10} \frac{1}{10} dx = \frac{1}{10}[x]_5^{10} = 0.5$$
  $$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X \leq 5) = 1 - F(5) = \\ = 1 - \frac{5}{10} = 0.5$$

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- exactly 7 minutes
  $P(X = 7) = 0$

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- the variance $D(X) = \frac{1}{12}(\beta - \alpha)^2 = \frac{1}{12}(10 - 0)^2 = \frac{100}{12} = 8.333$

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- the standard variation $\sigma = \sqrt{D(X)} = \sqrt{\frac{100}{12}} = 2.887$
- 90 % quantile $x_{0.90} = \alpha + 0.90(\beta - \alpha) = 0.9 \cdot 10 = 9$

The Exponential distribution

Definition

If the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{\delta} e^{-\frac{x-\alpha}{\delta}} & x > \alpha, \\ 0 & x \leq \alpha, \end{cases}$$

where $\alpha \in \mathbb{R}, \delta > 0$, then X is said to have an **exponential distribution** with parameters α and δ, written $X \sim Ex(\alpha, \delta)$.

The Exponential distribution

The cumulative distribution function is

$$F(x) = \begin{cases} 1 - e^{-\frac{x-\alpha}{\delta}} & x > \alpha, \\ 0 & x \leq \alpha. \end{cases}$$

The Exponential distribution

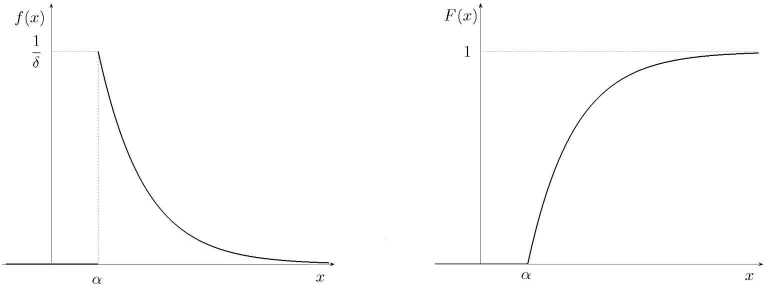

Figure: The probability density and the cumulative distribution function $Ex(\alpha, \delta)$

The Exponential distribution

The table summarizes some basic information about the exponential distribution.

$E(X)$	$D(X)$	$\alpha_3(X)$	$\alpha_4(X)$	quantiles x_P	$Me(X)$
$\alpha + \delta$	δ^2	2	6	$\alpha - \delta \ln(1 - P)$	$\alpha + \delta \ln 2$

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Examples: the queuing theory, the reliability theory, the renewal theory, a time we wait for service, a product lifetime, . . .

The Exponential distribution

Using:

- $P(X \leq x_0) = F(x_0) = 1 - e^{-\frac{x_0 - \alpha}{\delta}}$ for $x_0 > \alpha$

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- $P(X \leq x_0) = F(x_0) = 1 - e^{-\frac{x_0 - \alpha}{\delta}}$ for $x_0 > \alpha$
- $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) = e^{-\frac{x_1 - \alpha}{\delta}} - e^{-\frac{x_2 - \alpha}{\delta}}$ for $x_1, x_2 > \alpha$

Example

It has been found out the time we have to wait for a waiter is a random variable which has a exponential distribution with the mean 5 minutes and the standard deviation 2 minutes. Plot the probability density function and the distribution function. What is the probability that we will wait at most 5 minutes?

Example

The mean a the variance of the exponential distribution are $E(X) = \alpha + \delta$ and $D(X) = \delta^2$, thus

$$\begin{aligned} \alpha + \delta &= 5 \\ \delta &= 2 \end{aligned} \Rightarrow \alpha = 3, \delta = 2$$

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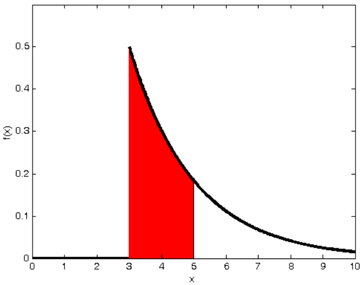
$$X \sim Ex(3, 2)$$

Example



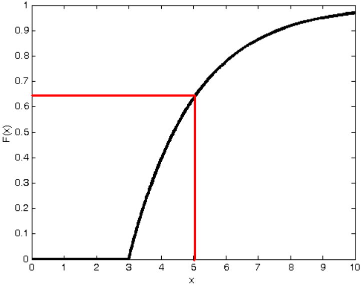


Figure: The probability density and the cumulative distribution function $Ex(\alpha, \delta)$

Example

The probability that we will wait at most 5 minutes is

$$P(X \leq 5) = F(5) = 1 - e^{-\frac{5-3}{2}} = 1 - e^{-1} = 0.632.$$

The Normal Distribution

Definition

If X has the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad x \in \mathbb{R}$$

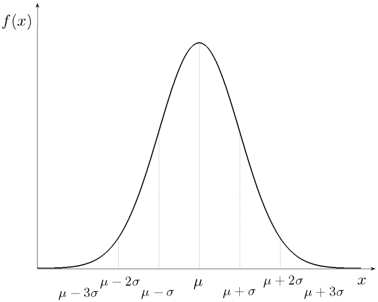
where $\mu \in \mathbb{R}, \sigma^2 > 0$, it is said to have a **normal distribution** with parameters μ and σ^2, written $X \sim N(\mu, \sigma^2)$.

The Normal Distribution

The cumulative distribution function is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \quad \text{for} \quad x \in \mathbb{R}$$

The Normal Distribution



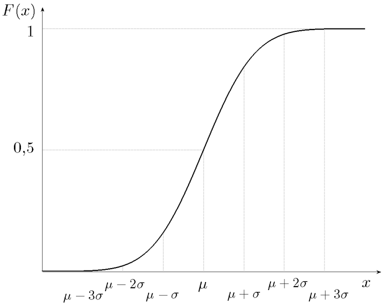


Figure: The probability density and the cumulative distribution function $N(\mu, \sigma^2)$

The Normal Distribution

The table summarizes some basic information about the normal distribution.

$E(X)$	$D(X)$	$\alpha_3(X)$	$\alpha_4(X)$	quantiles x_P	$Me(X)$	$Mo(X)$
μ	σ^2	0	0	$\mu + \sigma u_P$ [1]	μ	μ

[1] quantile of the standard normal distribution $N(0,1)$

The Normal Distribution

The normally distributed random variable fulfils :

- $P(\mu - \sigma < X < \mu + \sigma) = 0.683$

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- $P(\mu - \sigma < X < \mu + \sigma) = 0.683$
- $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.954$
- $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.997$

The Normal Distribution

Let us have the random variable $X \sim N(\mu, \sigma^2)$. The transformed random variable U

$$U = \frac{X - \mu}{\sigma}$$

has the normal distribution with the mean 0 and the variance 1 (the **standard normal distribution** $U \sim N(0, 1)$).

The Normal Distribution

The probability density function is

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \quad \text{for} \quad u \in \mathbb{R},$$

the cumulative distribution function is

$$\Phi(u) = \int\limits_{-\infty}^{u} \phi(t) dt = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{u} e^{-\frac{t^2}{2}} dt \quad \text{for} \quad u \in \mathbb{R}$$

The Normal Distribution

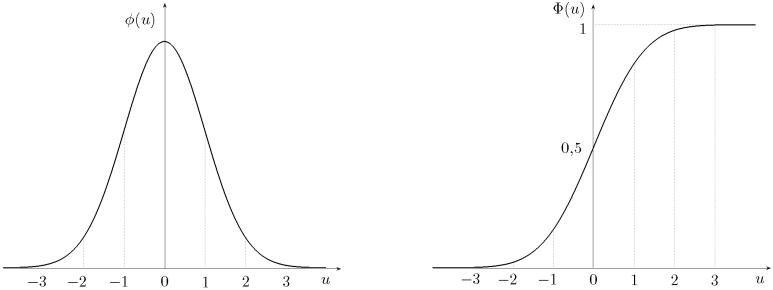

Figure: The probability density and the cumulative distribution function $N(0, 1)$

The Normal Distribution

The table summarizes some basic information about the standard normal distribution.

$E(X)$	$D(X)$	$\alpha_3(X)$	$\alpha_4(X)$	kvantily x_P	$Me(X)$	$Mo(X)$
0	1	0	0	u_P[1]	0	0

[1] the values are tabulated, for $P < 0.5$ is $u_P = -u_{1-P}$

The Normal Distribution

The values of the cumulative distribution function for positive values are tabulated, for negative values we can write

$$\Phi(-u) = 1 - \Phi(u).$$

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If $X \sim N(\mu, \sigma^2)$, $U \sim N(0,1)$, then the cumulative distribution function of the random variable X we can obtain using cumulative distribution function of U.

$$F(x_0) = P(X \leq x_0) = P(X - \mu \leq x_0 - \mu) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x_0-\mu}{\sigma}\right) =$$
$$= P\left(U \leq \frac{x_0-\mu}{\sigma}\right) = \Phi\left(\frac{x_0-\mu}{\sigma}\right)$$

The Normal Distribution

Quantiles of X (quantiles of U are tabulated):

$$F(x_P) = \Phi\left(\frac{x_P - \mu}{\sigma}\right) = \Phi(u_P),$$

thus

$$u_P = \frac{x_P - \mu}{\sigma} \Rightarrow x_P = \mu + \sigma u_P.$$

The Normal Distribution

Using:

- $P(X \leq x_0) = F(x_0) = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$

The Normal Distribution

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- $P(X \leq x_0) = F(x_0) = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$
- $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) = \Phi\left(\frac{x_2 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - \mu}{\sigma}\right)$

Example

During quality control we say that the component is acceptable if it's size is within the limits 26–27 mm. The size of the component has a normal distribution with the mean $\mu = 26.4\,\text{mm}$ and the standard deviation $\sigma = 0.2\,\text{mm}$. What is the probability that the size of the component is within the given limits?

Example

The random variable $X \sim N(26.4; 0.2^2)$.

$$\begin{aligned} P(26 \leq X \leq 27) &= F(27) - F(26) = \Phi\left(\tfrac{27-26.4}{0.2}\right) - \Phi\left(\tfrac{26-26.4}{0.2}\right) = \\ &= \Phi(3) - \Phi(-2) = \Phi(3) - (1 - \Phi(2)) = \\ &= 0.99865 - (1 - 0.97725) = 0.9759 \end{aligned}$$

Example

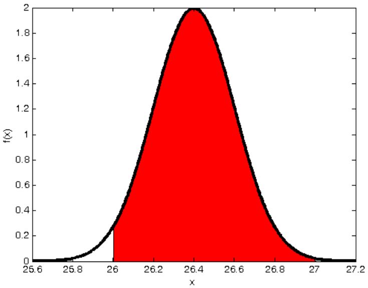

Figure: The probability density and the cumulative distribution function $N(26.4; 0.04)$

The Log-normal Distribution

Let us assume that X is a non-negative random variable. If a random variable $\ln X$ has a normal distribution $N(\mu, \sigma^2)$, then X has a log-normal distribution $LN(\mu, \sigma^2)$.

Definition

If the probability density function of X

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} & x > 0, \\ 0 & x \leq 0, \end{cases}$$

where $\mu \geq 0$, $\sigma > 0$, then X is said to have a **log-normal distribution** with parameters μ and σ^2, written $X \sim LN(\mu, \sigma^2)$.

The Log-normal Distribution

The table summarizes some basic information about the log-normal distribution.

$E(X)$	$D(X)$	$\alpha_3(X)$	$\alpha_4(X)$	quantiles x_P	$Mo(X)$
$e^{\mu+\sigma^2/2}$	$e^{2\mu}\omega(\omega-1)$	$\sqrt{\omega-1}(\omega+2)$	$\omega^4+2\omega^3+3\omega^2-6$	$e^{\mu+\sigma u_P}$	$e^{\mu-\sigma^2}$

where $\omega = e^{\sigma^2}$

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where $\omega = e^{\sigma^2}$

Examples: model of entry and wages distributions, a time of renewals, repairs, the theory of non-coherent particles, . . .

The Log-normal Distribution

If the random variable X has the log-normal distribution $X \sim LN(\mu, \sigma)$, then the transformed random variable

$$U = \frac{\ln X - \mu}{\sigma}$$

has the standard normal distribution $U \sim N(0, 1)$.

The Log-normal Distribution

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has the standard normal distribution $U \sim N(0, 1)$.
We can write

$$F(x_0) = \Phi\left(\frac{\ln x_0 - \mu}{\sigma}\right) = \Phi(u),$$

where $\Phi(u)$ is the cumulative distribution function $N(0, 1)$.

The Log-normal Distribution

Using:

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Example

We suppose that distance between vehicles on the highway (in seconds) is a random variable which is possible to describe by the log-normal distribution with parameters $\mu = 1.27$ a $\sigma^2 = 0.49$. What is the probability that the distance will be from 4 till 5 seconds?

Example

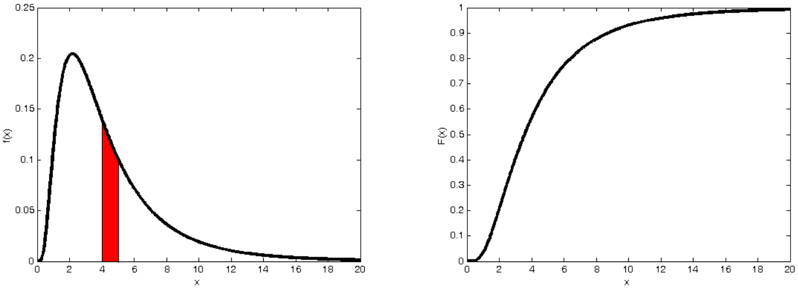

Figure: The probability density and the cumulative distribution function $LN(1.27; 0.7)$

Example

The probability is

$$P(4 \leq X \leq 5) = F(5) - F(4) = \Phi\left(\frac{\ln 5 - 1.27}{0.7}\right) - \Phi\left(\frac{\ln 4 - 1.27}{0.7}\right) = \\ = 0.68613 - 0.56597 = 0.12016$$

The Pearson χ^2 Distribution

Definition

Let us assume that $U_1, U_2, \ldots, U_\nu$ are independent normally distributed random variables ($N(0,1)$). The random variable

$$\chi^2 = U_1^2 + U_2^2 + \cdots + U_\nu^2,$$

has a χ^2-**distribution** with ν degrees of freedom.

The parameter ν (the number of freedom) usually represents the number of independent observation reduced by the number of linear conditions.

The Pearson χ^2 Distribution

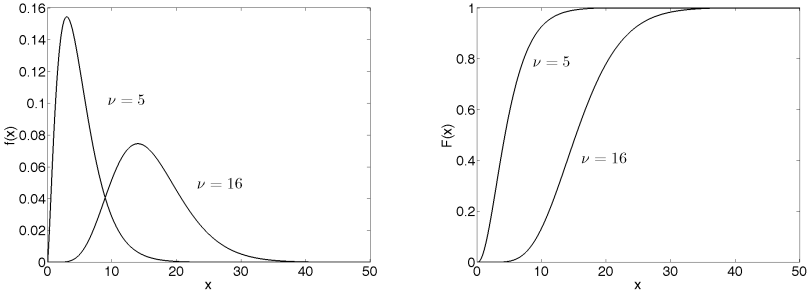

Figure: The probability density and the cumulative distribution function $\chi^2(5)$ and $\chi^2(16)$

The Pearson χ^2 Distribution

In the future the quantiles of the χ^2 distribution will be useful. They are usually tabulated for various values P and degrees of freedom $\nu \leq 30$. For $\nu > 30$ is possible to use an approximation

$$\chi^2_P(\nu) \approx \frac{1}{2}\left(\sqrt{2\nu - 1} + u_P\right)^2,$$

where u_P is the quantile of $N(0, 1)$.

The Student distribution $t(\nu)$

Definition

If a random variable U has a standard normal distribution $U \sim N(0, 1)$, a random variable χ^2 has a Pearson distribution $\chi^2 \sim \chi^2(\nu)$ and if U and χ^2 are independent, then a random variable

$$t = \frac{U}{\sqrt{\frac{\chi^2}{\nu}}}$$

has a **Student distribution** with ν degrees of freedom, written $t \sim t(\nu)$.

The Student distribution $t(\nu)$

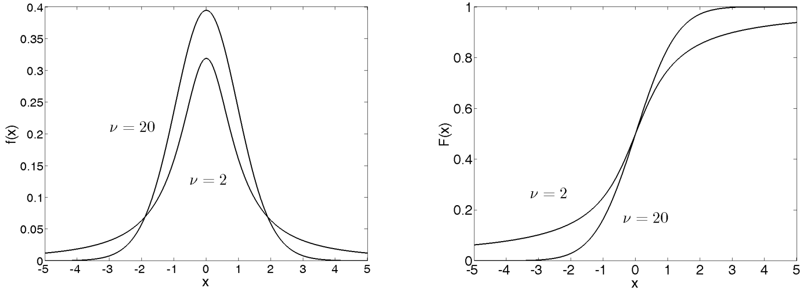

Figure: The probability density and the cumulative distribution function $t(2)$ and $t(20)$

The Student distribution $t(\nu)$

The probability density function is symmetric with the mean $E(t) = 0$. Quantiles of the Student distribution are tabulated for $\nu \leq 30$ and $P > 0{,}5$, for $P < 0,5$ is

$$t_P = -t_{1-P}.$$

Whether $\nu > 30$, we can use an approximation

$$t_p \approx u_P.$$

The Fisher-Snedecor Distribution $F(\nu_1, \nu_2)$

Definition

If a random variable χ_1^2 has $\chi_1^2 \sim \chi^2(\nu_1)$ with ν_1 degrees of freedom and a random variable χ_2^2 has $\chi_2^2 \sim \chi^2(\nu_2)$ with ν_2 degrees of freedom and they are independent, then a random variable

$$F = \frac{\chi_1^2}{\nu_1} : \frac{\chi_2^2}{\nu_2}$$

has a **Fisher-Snedecor** distribution with ν_1 and ν_2 degrees of freedom, written $F \sim F(\nu_1, \nu_2)$.

The Fisher-Snedecor Distribution $F(\nu_1, \nu_2)$

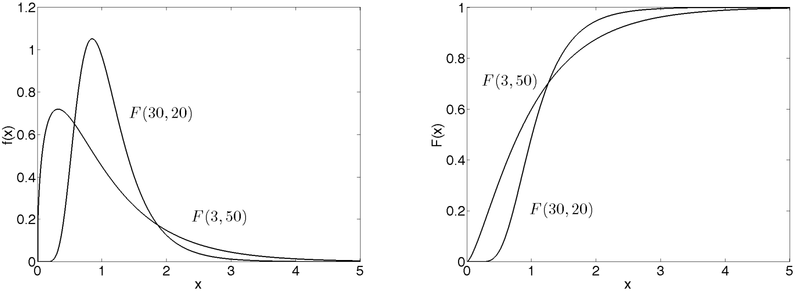

Figure: The probability density and the cumulative distribution function $F(30, 20)$ and $F(3, 50)$

The Fisher-Snedecor Distribution $F(\nu_1, \nu_2)$

The Fischer-Snedecor distribution is asymmetric.
Quantiles of F distribution are tabulated for $P > 0{,}5$, for $P < 0{,}5$ we can use the formula

$$F_P(\nu_1, \nu_2) = \frac{1}{F_{1-P}(\nu_2, \nu_1)}.$$