1 RANDOM VARIABLE

1.1 Random variable

1.1.1 Exercises 1.1

1. Decide, if it is a discrete or continuous random variable and determine the range regarding
   a) the sum of outcomes corresponding to throwing 3 fair dies,
   b) the waiting for a tram which goes regularly each 5 minutes,
   c) the number of germinated seeds from total number 50,
   d) the number of customers of given petrol station,
   e) the performance in 400-metre run,
   f) the pollutant concentration (percentage).
2. Give examples of discrete or continuous random variables regarding
   a) $(0, \infty)$,
   b) $1, 2, \dots,$
   c) $\langle 0, 30)$,
   d) $0, 1, 2, \dots, n$, where n is a natural number.

Solution.
1. a) discrete $M = \{3, 4, 5, \dots, 18\}$; b) continuous $M = \langle 0, 5)$; c) discrete $M = \{0, 1, 2, \dots, 50\}$; d) discrete $M = \{0, 1, 2, \dots, \}$; e) continuous $M = (0, \infty)$; f) continuous $M = (0, 100)$.
2. a) e.g. the performance in putting the shot; b) e.g. the number of repetitions of throwing a fair die with the favourable outcome 6; c) e.g. the waiting for a bus which goes regularly each 30 minutes; d) e.g. the number of tosses a coin with favourable outcome „face“ and the total number of repetitions n.

1.2 Distribution function

1.3 Probability function

1.3.1 Exercises 1.3

1. The probability function of X is defined by $p(x) = 0.02\ 0.07\ \ 0.18\ 0.25\ 0.30\ 0.18$ for $x = -1, 0, 1, 2, 3, 4$ and otherwise $p(x) = 0$.
   a) Determine the distribution function. Draw the graph of the probability function and the graph of the distribution function.
   b) Compute probabilities $P(X < 3)$, $P(X \geq 0)$ and $P(0 \leq X < 4)$.

2. The distribution function of X is defined by

$$F(x)=\begin{cases} 0 & \text{for} \quad x<0, \\ 0.125 & \quad 0\le x<1, \\ 0.5 & \quad 1\le x<2, \\ 0.875 & \quad 2\le x<3, \\ 1 & \quad x\ge 3. \end{cases}$$

   a) Determine the probability function. Draw the graph of the probability function and the graph of the distribution function.
   b) Compute probabilities $P(X<2)$, $P(X\ge 1)$ and $P(1\le X<3)$.
3. A player throws a fair die three times. The random variable X represents the number of outcomes 6.
   a) Describe this random variable X by the way of the probability function and the distribution function, draw graphs.
   b) What is the probability that the outcome 6 occurs at least once?
4. A shooter shoots at a target five times. Each successful hit is evaluated by 3 points, otherwise -1 point. The probability of correct hits is for each shoot 2/3. Determine the probability function of the number of points.
5. Let $p(x)$ is a function defined by

$$p(x)=\begin{cases} k\cdot 0.4^x & \text{for} \quad x=1,2,3,\dots, \\ 0 & \quad \text{otherwise.} \end{cases}$$

   a) Determine the constant $k\in\mathbb{R}$ so that $p(x)$ is the probability function of the random variable X.
   b) Compute probabilities $P(X<4)$, $P(X\ge 5)$, $P(-1<X\le 2)$.

Solution.

1. a) 0 for $x<-1$; 0.02 for $-1\le x<0$; 0.09 for $0\le x<1$; 0.27 for $1\le x<2$; 0.52 for $2\le x<3$; 0.82 for $3\le x<4$; 1 for $x\ge 4$; b) 0.52; 0.98; 0.80;
2. a) 0.125 for $x=0$; 0.375 for $x=1$; 0.375 for $x=2$; 0.125 for $x=3$; 0 otherwise; b) 0.5; 0.875; 0.75;
3. a) $p(x)$: 125/216 for $x=0$, 25/72 for $x=1$, 5/72 for $x=2$ a 1/216 for $x=3$, 0 otherwise; $F(x)$: 0 for $x<0$, 125/216 for $0\le x<1$, 25/27 for $1\le x<2$, 215/216 for $2\le x<3$ a 1 for $x\ge 3$; b) 91/216;
4. $p(x)$: 1/243 for $x=-5$, 10/243 for $x=-1$, 40/243 for $x=3$, 80/243 for $x=7$, 80/243 for $x=11$, 32/243 for $x=15$;
5. a) 3/2; b) 0.936; 0.0256; 0.84.

1.4 Probability density function

1.4.1 Exercises 1.4

1. The distribution function of a random variable X is defined by

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leq 1, \\ \frac{x-1}{4} & \quad 1 < x < 5, \\ 1 & \quad x \geq 5. \end{cases}$$

   a) Determine the probability density function and draw graphs of both functions.
   b) Compute probabilities $P(X < 3)$, $P(2 \leq X < 4)$, $P(0 < X < 2)$ and $P(X = 3)$.

2. The probability density function of a random variable X is defined by

$$f(x) = \begin{cases} c - 2x & \text{for} \quad 0 < x < 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

   a) Determine the constant c a draw the graph of the function $f(x)$.
   b) Determine the distribution function and draw the graph.
   c) Compute probabilities $P(X \geq 0.5)$ and $P(0 < X \leq 0.75)$.

3. The probability density function of a random variable X is defined by

$$f(x) = \begin{cases} \frac{c}{x^4} & \text{for} \quad x > 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

   a) Determine the constant c a draw the graph of the function $f(x)$.
   b) Determine the distribution function and draw the graph.
   c) Compute probabilities $P(X \geq 2)$, $P(X > 1.5)$ and $P(2 < X \leq 3)$.

4. The distribution function of Rayleigh distribution is

$$F(x) = \begin{cases} C - e^{-\frac{x^2}{2\sigma^2}} & \text{for} \quad x > 0, \\ 0 & \quad x \leq 0. \end{cases}$$

   Determine the constant $C \in \mathbb{R}$ and the probability density function $f(x)$.

5. A random variable X has distribution described by the distribution function

$$F(x) = \begin{cases} 1 - e^{-\frac{x-1}{2}} & \text{for} \quad x > 1, \\ 0 & \quad x \leq 1. \end{cases}$$

   a) Determine the density function and draw the graph.
   b) Compute probabilities $P(X < 2)$, $P(1 < X < 3)$, $P(x > 4)$.

Solution.
1. a) $f(x)$: $1/4$ for $1 < x < 5$, 0 otherwise; b) 0.5; 0.5; 0.25; 0;
2. a) $c = 2$; b) $F(x)$: 0 for $x \leq 0$, $2x - x^2$ for $0 < x < 1$, 1 for $x \geq 1$; c) 0.25; 0.9375;
3. a) $c = 3$; b) $F(x)$: 0 for $x \leq 1$, $1 - 1/x^3$ for $x > 1$; c) 0.125; 0.296; 0.088;
4. $C = 1$, $f(x)$: 0 for $x \leq 0$, $\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$ for $x > 0$;
5. a) $f(x)$: $\frac{1}{2} e^{-\frac{x-1}{2}}$ for $x > 1$, 0 for $x \leq 1$; b) 0.393; 0.632; 0.223.

1.5 Measure of location

1.5.1 Exercises 1.5

1. Determine the mean and the mode of a random variable X which is defined by the probability function:

x	-1	0	1	2	3	4
$p(x)$	0.02	0.07	0.18	0.25	0.30	0.18

2. Compute the mean of a random variable which determines the outcome (number) of rolling a fair die.
3. A shooter shoots at a target four times. The probability of a correct hit is for each shoot 0.8. Compute the mean and the mode of a random variable X which determines the number of correct hits.
4. Compute the mean, the median and the upper decile (quantile $x_{0.90}$) of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{for} \quad 0 < x < 1, \\ 0 & \quad\text{otherwise.} \end{cases}$$

5. Compute the mean, the median and the upper quartile (quantile $x_{0.75}$) of a random variable X which is defined by the distribution function:

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leq 0, \\ x^2 & \quad 0 < x < 1, \\ 1 & \quad x \geq 1. \end{cases}$$

6. Compute the mean, the median and the mode of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{for} \quad -1 < x < 1, \\ 0 & \quad\text{otherwise.} \end{cases}$$

Solution.
1. 2.28; 3;
2. 3.5;
3. 3.2; 3 and 4;
4. 0.75; 0.794; 0.965;
5. 0.667; 0.707; 0.866;
6. 0; 0; 0.

1.6 Measure of dispersion

1.6.1 Exercises 1.6

It is possible to use results from 1.5.1 in next exercises.

1. Determine the variance and the standard deviation of a random variable X which is defined by the probability function (see 1.3.1 exercise 1):

x	-1	0	1	2	3	4
$p(x)$	0.02	0.07	0.18	0.25	0.30	0.18

2. Compute the variance and the standard deviation of a random variable X which determines the outcome (number) of rolling a fair die.
3. A shooter shoots at a target four times. The probability of a correct hit is for each shoot 0.8. Compute the variance and the standard deviation of a random variable X which determines the number of correct hits.
4. Compute the variance and the standard deviation of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{for} \quad 0 < x < 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

5. Compute the variance and the standard deviation of a random variable X which is defined by the distribution function:

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leq 0, \\ x^2 & \quad 0 < x < 1, \\ 1 & \quad x \geq 1. \end{cases}$$

6. Compute the variance and the standard deviation of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{for} \quad -1 < x < 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

Solution.
1. 1.582; 1.258;
2. 2.917; 1.708;
3. 0.64; 0.8;
4. 0.038; 0.194;
5. 0.056; 0.236;
6. 0.2; 0.447.

1.7 Measure of concentration

1.7.1 Exercises 1.7

It is possible to use results from 1.5.1 and 1.6.1 in next exercises.

1. Determine the skewness and the kurtosis of a random variable X which is defined by the probability function

x	-1	0	1	2	3	4
$p(x)$	0.02	0.07	0.18	0.25	0.30	0.18

2. Compute the skewness and the kurtosis of a random variable X which determines the outcome (number) of rolling a fair die.
3. A shooter shoots at a target four times. The probability of a correct hit is for each shoot 0.8. Compute the skewness and the kurtosis of a random variable X which determines the number of correct hits.
4. Compute the skewness and the kurtosis of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{for} \quad 0 < x < 1, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

5. Compute the skewness and the kurtosis of a random variable X which is defined by the distribution function:

$$F(x) = \begin{cases} 0 & \text{for} \quad x \le 0, \\ x^2 & \quad\quad 0 < x < 1, \\ 1 & \quad\quad x \ge 1. \end{cases}$$

6. Compute the skewness and the kurtosis of a random variable X which is defined by the probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{for} \quad -1 < x < 1, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

Solution.
1. -0.448; -0.463;
2. 0; -1.960;
3. 0.75; 0.0625;
4. -0.861; 0.095;
5. -0.566; -0.6;
6. 0; -0.857.